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# On the Maximum Number of Switches Between Two Production Systems\*

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## Introduction

In his "Production of Commodities by Means of Commodities"<sup>1</sup> Piero Sraffa introduces a classification of commodities into 'basics' and 'nonbasics': A commodity is called 'basic' to the system of production<sup>2</sup> if it enters, directly or indirectly, into every other commodity in the system; a commodity which does not do so is called 'nonbasic'. Opinions have diverged as to the role and significance of this distinction. In the literature, production systems containing only basics are more in vogue<sup>3</sup>. In an article on Sraffa's book, Professor Newman<sup>4</sup> proposes to 'abandon' nonbasics, by omitting them from the system, in the interest of analytical simplicity; particularly, to avoid the possibility of negative prices which a system containing nonbasics may give rise to (see Appendix below). However the existence of nonbasics is an objective property of a system and while nonbasics may be ignored in a first approximation they will have ultimately to be taken into account. Other writers, while considering nonbasics, have adopted, instead of this distinction between commodities,

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<sup>1</sup> Sraffa P., Production of Commodities by Means of Commodities, Cambridge University Press, 1960.

<sup>2</sup> A system of production (alternatively, a production system) producing n commodities is a set of n production methods, one for each and each producing a single commodity.

<sup>3</sup> The 'Leontief system' which is frequently used is one such system.

<sup>4</sup> Newman P., Production of Commodities by Means of Commodities, Schweizerische Zeitschrift für Volkswirtschaft und Statistik 1962, p. 58–75.

another classification based on an abstract property of the 'technology matrix' as being 'decomposable' or 'indecomposable'. A matrix is called 'decomposable' when by suitable interchange of columns and rows it can be reduced to the form  $\begin{bmatrix} A_1 A_2 \\ 0 & A_3 \end{bmatrix}$  where  $A_2$  and  $A_3$  are square matrices and 0 is a zero matrix. A matrix which cannot be so reduced is called 'indecomposable'. Thus a system containing at least one nonbasic has a decomposable technology matrix while a system with all basics has an indecomposable one. However, as will be seen, the basic-nonbasic distinction, referring to commodities in a given economic system, uses directly more information about it and does so in a way that helps perceive the economic content of the distinction.

This paper discusses a problem relating to 'reswitching of methods of production', an issue at the centre of a current controversy. A 'reswitch' in the methods of production is said to occur when, of two methods of production, one which has ceased to be the more profitable because of a change in the rate of profit becomes again more profitable than the other as the rate of profit moves further in the same direction. We shall take up here the more specific question of the maximum number of switches between two production systems and incidentally note how part of the difficulty in the reswitching controversy arises from not taking into account the particular role played by nonbasics.

We consider a situation where a number of commodities are being produced in annual cycles. Each commodity is produced by a separate industry, i.e. there is no joint production. A system of production with a specified net output is composed of the methods of production for the commodities that form the net output as well as others which enter, directly or indirectly, into their production. There is one method for each commodity in a system. We then suppose that there is an alternative method for one of the commodities and an alternative system is formed characterised by the use of the alternative method for that commodity. The introduction of the alternative method could entail the use of new commodities while possibly dropping some others. The switch point between the two systems corresponds to the rate of profit at which the two alternative methods produce the commodity at the same price (that is, at the switch point, the wage rate as also the prices of the commodities produced in both systems are equal). We deal in Section I with the case where each system consists only of commodities basic to it. In Section II we take up the case where the two systems also include nonbasics and where they differ in the method for a commodity which is nonbasic to both. We also examine there whether nonbasics entering the value unit (in terms of which wages and prices are expressed) but not entering either of the alternative methods influence the number of switching possibilities, as is sometimes im-

plied (see p. 422 below). A question that arises here concerns the conditions guaranteeing the positivity of prices in a system including nonbasics. In this connection, we have reproduced, in the Appendix, letters<sup>5</sup> that were exchanged between *Sraffa* and *Newman*, following *Newman's* article, referred to above, which throw light on this point. In Section III we consider the advantages of the basic-nonbasic distinction for the discussion of switching. Section IV contains the conclusions.

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## Production Systems Consisting of only Basics

Consider a production system A involving m commodities, all basics. Suppose that for one of the commodities an alternative method of production is known which entails the use of some new commodities while possibly dropping out some others. Suppose the alternative production system formed by replacing the former method by the latter, call it system B, has n commodities all basics to it and that the two systems A and B have s commodities common to both; so that there are (m-s) commodities used exclusively in system A and (n-s) commodities used exclusively in system B. For convenience we renumber the commodities so that  $1, 2 \ldots s$  are the s commodities common to the two systems, s+1,  $s+2 \ldots m$  are the (m-s) commodities exclusive to system B. A switch point from one system to the other would be found at the rate of profit at which the wage and the price of each of the s common commodities are equal in the two systems.

Assuming wages are paid at the end of each annual cycle we write the price equations for the two systems:

System A

<sup>5</sup> I am grateful to Professor *Newman* and Mr. *Sraffa* for allowing me to publish these letters. 411

where  $p_{1a}, p_{2a} \ldots p_{ma}$  are the prices of commodities  $1, 2 \ldots m$  and  $W_a$  the wage rate in system  $\mathcal{A}$  and  $\lambda = 1 + r$  where r is the rate of profit;  $a_{ij}$   $(i, j = 1, 2, \ldots, m)$  and  $a_{0j}$   $(j = 1, 2 \ldots, m)$  are the commodity input and labour coefficients respectively for System  $\mathcal{A}$ . An analogous notation to represent prices and the wage rate in the system B is adopted to write the price equations for the system B. (It would be noted that  $i, j = 1, 2 \ldots s, m+1, \ldots, m+n-s$ and  $a_{ij} = b_{ij}$  for  $j = 1, 2 \ldots s-1$ .)

We need now to find out such values of  $\lambda$  at which  $p_{ia} = p_{ib} (i = 1, 2 \dots s)$ when  $W_a = W_b = W^6$ . We take  $p_{1a} = p_{1b} = 1$  so that commodity 1 is chosen as numeraire. We have the problem of unequal numbers and different kinds of basics in the two systems (namely commodities  $1, 2 \dots m$  in system A and  $1, \dots, s, s + 1, \dots, m + n - s$  in system B) which affect the wage profit relations in the respective systems. We now introduce in system A, as nonbasics, the

<sup>6</sup> The condition regarding the equality of relative prices of the *s* commodities common to the two systems is equivalent to stating that the wage in terms of any of them should be equal, at the switch point, in the two systems. A point of some interest to note is that, if we consider any two production systems differing in the method of production for more than one commodity common to them and express the wage and prices in the two systems in terms of someone of the commodities common to them, the relative prices for these common commodities in the two systems may not necessarily be equal at all the intersections of the wage profit curves for the two systems. The equality of the relative prices would have to be laid down as *a priori* condition to obtain the switch points among those points of intersections.

commodities which are exclusive to system B and thereby augment the matrix <sup>7</sup>; let the matrix so augmented be called system  $A^+$ . Similarly we construct  $B^+$  from system B. These would appear as follows:

$$\begin{bmatrix} A^{+}, A_{0}^{+} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{s1} \\ a_{12} & a_{22} & \cdots & a_{s2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1s-1} & a_{2s-1} & \cdots & a_{ss-1} \end{bmatrix} \qquad \bigcirc \qquad \bigcirc \qquad \bigcirc \qquad \bigcirc \qquad \begin{bmatrix} a_{01} \\ a_{02} \\ \vdots \\ a_{0s-1} \\ a_{0s-1} \end{bmatrix}$$

$$\begin{bmatrix} a_{1s} & a_{2s} & \cdots & a_{ss} \\ a_{1s} & a_{2s} & \cdots & a_{ss} \end{bmatrix} \begin{bmatrix} a_{s+1s} & \cdots & a_{ms} \\ \vdots & \vdots & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{sm} \end{bmatrix} \begin{bmatrix} a_{s+1s+1} & \cdots & a_{mm} \\ a_{s+1m} & \cdots & a_{mm} \end{bmatrix} \bigcirc \qquad \begin{bmatrix} a_{0s+1} \\ \vdots \\ a_{0m} \end{bmatrix}$$

$$\begin{bmatrix} b_{1m+1} & b_{2m+1} & \cdots & b_{sm+1} \\ \vdots & \vdots & \vdots \\ b_{1m+n-s} & b_{2m+n-s} & \cdots & b_{sm+n-s} \end{bmatrix} \bigcirc \qquad \begin{bmatrix} b_{m+1m+1} & b_{m+n-sm+1} \\ \vdots \\ b_{m+1m+n-s} & b_{m+n-sm+n-s} \end{bmatrix} \begin{bmatrix} b_{0m+1} \\ \vdots \\ b_{0m+n-s} \end{bmatrix}$$

 $[B^+, B_0^+] =$ 

<i>a</i> <sub>11</sub>	a <sub>21</sub>	a <sub>si</sub>			a <sub>01</sub>
a <sub>12</sub>	a <sub>22</sub>	a <sub>s2</sub>			a <sub>02</sub>
÷	:	• •	0	0	
a <sub>1s-1</sub>	a <sub>2s-1</sub>	a <sub>ss-1</sub>			a <sub>0s-1</sub>
b <sub>1s</sub>	b <sub>2s</sub>	b <sub>ss</sub>	0	$b_{m+1s} \ldots b_{m+n-ss}$	b <sub>0s</sub>
a <sub>is+1</sub>	a <sub>2s+1</sub>	a <sub>ss+1</sub>	$a_{s+1s+1} \ldots a_{ms+1}$		a <sub>0s+1</sub>
÷	:			0	:
<i>a</i> <sub>im</sub>	$a_{2m}$	a <sub>sm</sub>	$a_{s+1m} \ldots a_{mm}$		a <sub>0m</sub>
$b_{1m+1}$	$b_{2m+1}$	b <sub>sm+1</sub>		$b_{m+1m+1}$ $b_{m+n-sm+1}$	<i>b</i> <sub>0<i>m</i>+1</sub>
	:	:	0		÷
b <sub>1m+n-1</sub>	$b_{2m+n-1}$	$b_{sm+n-s}$		$b_{m+1m+n-s}b_{m+n-sm+n-s}$	$b_{0m+n-s}$

 $^{7}$  We do so since, at a switch point, all the methods in both systems must be competitive.

where  $\mathcal{A}^+$  and  $\mathcal{B}^+$  are augmented matrices of commodity input coefficients and  $\mathcal{A}_0^+$  and  $\mathcal{B}_0^+$  are augmented matrices of labour coefficients. The switch point values of  $\lambda$  which satisfy the condition  $p_{ia}^+ = p_{ib}^+ = p_i^+$  and  $W_a^+ = W_b^+$  $= W^+$  are to be obtained by solving a vector of polynomial equations given by

$$[F_{i}^{+}(\lambda)] = [A^{+}-B^{+}] \lambda [p^{+}] + [A_{0}^{+}-B_{0}^{+}] W^{+} = 0$$
(1)  
$$i = 1, 2 \dots m + n - s.$$

where matrices  $A^+$  and  $B^+$  are both  $(m+n-s) \times (m+n-s)$  and  $p^+ = (1, p_2^+, p_3^+ \dots p_{m+n-s}^+)$ 

There would be as many non-zero elements in this vector as the number of differing methods of production in the augmented systems and in this case, therefore, there is only one polynomial to be solved, namely,  $F_{(s)}(\lambda) = 0$ . In other words, it is sufficient to equate the price of the commodity *s*, the only commodity to have a different method of production in the two systems. We could solve for  $p^+$  and  $W^+$  in terms of  $\lambda$  in either of the two systems  $\mathcal{A}^+$  or  $B^{+8}$ .

From the system  $A^+$ , we can write:

$$p^{+} = [p_{ia}^{+}] = \frac{[J_{i}^{+}(\lambda)]}{[g^{+}(\lambda)]} (2) \text{ and } \mathcal{W}_{a}^{+} = \mathcal{W}_{b}^{+} = \frac{f^{+}(\lambda)}{g^{+}(\lambda)}$$
(3)

where  $J_{(i)}^+(\lambda)$ ,  $f^+(\lambda)$  and  $g^+(\lambda)$  are of at most (m+n-s-1), (m+n-s) and (m+n-s-1) degree in  $\lambda$  respectively. Hence the polynomial function in (1) can have at most (m+n-s) roots and hence the maximum number of switches between the two systems is (m+n-s). As a polar case, if the two systems A and B have only one basic commodity common between them, the maximum number of switch points would be  $(m+n-1)^9$ .

Of the number of possible switches thus obtained as an upper bound, the economically relevant number of switch points would be obtained only after excluding repeated counting of repeated roots, complex roots and those lying beyond the range  $1 \leq \lambda \leq 1 + R$  where R is the lower of the two maximum rates of profit for the two systems<sup>10</sup>. Also, we consider only those situations in which all prices are positive<sup>11</sup>.

<sup>8</sup> Since  $p_{ia}^+ = p_{ib}^+ = p_i^+$  and  $W_a^+ = W_b^+$  either of the two systems can be so used.

<sup>9</sup> A particular illustration of this is the result obtained by Joan Robinson and K. A. Naqvi (Quarterly Journal of Economics., p. 590) where they consider two production systems, one with wheat and iron as basics and another with wheat and aluminium as basics and obtain three switch points between them.

<sup>10</sup> These switch points could also include such cases where, at the switch point rate of profit, the wage-profit curve for one system is tangential to that of the other wholly from above: that is, the same system continues to be the more profitable one on both sides of the switch point.

414

<sup>11</sup> See p. 418–9 below.

The case of the same basics in the two alternative systems which is often treated as 'general'<sup>12</sup> is seen to be only a particular case of the above more general formulation. With all n basics common to the two systems the maximum number of switches is seen to be only n. The assumption that the two systems have the same numbers and kinds of basics while they differ in the methods of production is extremeley restrictive since it is unlikely that two different methods will use identical materials and tools.

The discussion of switching possibilities between two production systems has been usually conducted under the assumption that the two systems under consideration may differ in the methods of production for more than one (and up to all) commodities common to them. This has been described as the most general model; but, far from being a general case this would be a very exceptional one<sup>13</sup>. Switches would occur between systems differing in the method of production for only one basic commodity common to the two systems. When more than one basic common to them is produced by a different method in the two systems, it is clear from the condition for obtaining the switch point set out in (1) above that a set of polynomial functions in  $\lambda$  of that number would have to have at least one common root. Such a condition would be fulfilled only as a fluke<sup>14</sup>.

<sup>12</sup> See, for example, Bruno M., Burmeister E. and Sheshinski E., The nature and implications of reswitching of techniques, Quarterly Journal of Economics 1966, p. 526-553.

<sup>13</sup> Analytically there is no loss of generality involved in a procedure of successive consideration of production systems using a different method of production for only one of the commodities common to them as, given all possible systems of production, it could not lead to any different outermost boundary of wage-profit curves. Incidentally, it would be noted that whatever be the number of commodities produced by different methods in the two systems the *maximum* number of switching possibilities would still be equal to the total number of distinct (without double counting) basics in the two systems together.

<sup>14</sup> Alternatively, we could arrive at the same conclusion by observing that a system with m basic commodities (such as A above) has (m + 1) unknowns (m-1 relative prices, wage and the rate of profit) and m independent equations to solve them. Hence one more additional equation can be accommodated to make the system determinate even though it does not bring in any additional commodity with its price. This additional equation would be the alternative method for one commodity in the system. If the alternative method brings in additional commodities there would have to be as many additional price equations. In our example above there are (m + n - s) distinct commodities in the two systems together and (m + n - s + 1) independent methods would be needed to determine the prices, wage rate and the rate of profit. If more than one commodity in system A has a different method in system B the system of equations would be overdetermined.

### Systems Including Commodities which are Nonbasics to Both

The production systems discussed so far involved commodities which were basic to one or the other system. We now turn to those which include commodities that are nonbasics to both. Even in such systems, when the two systems are characterised by a different method of production only for a *basic* produced in both, the switch points between the two methods for the basic (and hence the two systems) would be determined by solving for prices within the augmented systems; the latter would include *only* the methods of production for the commodities which are basics to one or the other system. The methods of production for commodities which are nonbasic to both systems can be ignored. We take up two cases where the methods of production of such nonbasics may not be so ignored.

1. If there are alternative methods of production for a nonbasic there would be switches in the method of production for that nonbasic as the rate of profit changes<sup>15</sup>. (Each of the basics has only one known method.)

2. Nonbasics may enter the value unit in terms of which prices and wage are expressed. It is evident that the nonbasics could enter the price equations of the basics only indirectly in this way. If prices are expressed as functions of the rate of profit, the maximum degree of the price equation for a basic would be given by the number of commodities entering directly or indirectly into this value unit which includes nonbasics. This seems to have suggested that the maximum number of possible switches is also given by that number (see p.418 below). We examine the question whether the nonbasics which enter the value unit, directly or indirectly, but do not enter, directly or indirectly, into either of the alternative methods between which switches are being considered, influence the switching possibilities. We first consider the question of the alternative methods for a nonbasic.

Alternative Methods for a Nonbasic: Suppose that one of the nonbasics in a system A has an alternative method of production. When the alternative method is used, it might entail the use of some nonbasics peculiar to itself while possibly dropping some others. Let us call the system characterised by the use of the latter method for the nonbasic system B. Suppose also that commodity 1 (basic to both systems) is numeraire. We can follow the same procedure as on page 413 above and write the augmented systems  $A^+$  and  $B^+$  which would now include, in addition to the methods of production for com-

<sup>15</sup> However the switches in the method of a nonbasic have to be clearly distinguished from those for a basic inasmuch as the former would not affect the relative prices of the basics in the system or the maximum rate of profit whereas the latter do.

modities basic to at least one system also those for nonbasics to both which enter, directly or indirectly, into one or the other of the alternative methods of production for the nonbasic in question. Such commodities as are nonbasics to both systems and do not enter either of these alternative methods would not appear in the augmented systems. The augmented matrices would differ in only one row, namely that representing the method of production for the nonbasic with the alternative methods. The maximum number of switches for the two systems is given, as in the earlier case, by the dimension of the augmented matrix, i.e. by the number of distinct commodities, basic and nonbasic, without double counting, that enter in at least one of the alternative methods of production for the nonbasic<sup>16</sup>.

Nonbasic Entering Value Unit. Suppose that there are alternative methods of production for only a basic and nonbasics enter the value unit. Further that these are nonbasics to both the systems, the two systems differing in the method for the basic<sup>17</sup>. Would the nonbasics entering the value unit affect the maximum number of switches between the two systems?

As a simple illustration we take production systems  $\mathcal{A}$  and B each with two basics (designated commodities 1 and 2 in system  $\mathcal{A}$  and 1 and 4 in system B) and one nonbasic (commodity 3 in both). They differ in the method for commodity 1. The nonbasic is produced by the same method in the two systems and forms the value unit. Representing the two systems as below:

System A

$$\begin{aligned} & (a_{11} p_{1a} + a_{21} p_{2a}) \lambda + a_{01} W_a = p_{1a} \\ & (a_{12} p_{1a} + a_{22} p_{2a}) \lambda + a_{02} W_a = p_{2a} \\ & (a_{13} p_{1a} + a_{33} p_{3a}) \lambda + a_{03} W_a = p_{3a} \end{aligned}$$

<sup>16</sup> One may consider, as a curiosum, the case of a commodity basic to system A which when produced by an alternative method becomes itself a nonbasic in system B, each of the other commodities having only one known method. This would, however, imply that the two systems would have no commodities in common which are basic to both.

<sup>17</sup> This needs some clarification: Two production systems differing in the method of production for one of the basics common to them could have one or more commodities which are basic to one and not to the other. If wages and prices are expressed in terms of the 'standard commodity' of either one of the systems (for the definition of the 'standard commodity' see *Sraffa P.*, op. cit., p. 18–20) as *Sraffa* does (see op. cit., p. 85) or in terms of any value unit involving commodities exclusively basic to one of the systems, the other system will have its prices and wage expressed in terms of a value unit involving nonbasics to itself. These nonbasics are however basics to the other system and hence enter, directly or indirectly, into the production of the basic (with alternative methods) in that system.

System B

With  $P_{3a} = P_{3b} = 1$ , the switch points are to be obtained by augmenting A and B to  $A^+$  and  $B^+$  respectively as discussed earlier and by solving the following polynomial:

$$(a_{11}-b_{11}) \lambda p_{1a}^+ + a_{21} \lambda p_{2a}^+ + (-b_{41}) \lambda p_{4a}^+ + (a_{01}-b_{01}) \mathcal{W}_a^+ = 0$$
(4)

where  $P_{1a}^+$ ,  $P_{2a}^+$ ,  $P_{4a}^+$  and  $W_a^+$  are themselves polynomials in  $\lambda$ .

In this simple case we can make the following observations:

1. If  $a_{33} = 0$ , the nonbasic does not use itself in its own production, the polynomial in  $\lambda$  in (4) above has the maximum degree only three and hence the maximum number of switch points is only three.

2. If  $a_{33} \neq 0$  then  $\lambda = 1/a_{33}$  happens to be the additional solution for  $\lambda$ . It would be noted, however, that at this value of  $\lambda$ , with the nonbasic as the value unit, the prices of the basic commodities can no more satisfy the positivity condition<sup>18</sup>.

The observations hold even if a composite commodity consisting of basics and nonbasics (e.g.  $q_1 p_{1a} + q_2 p_{2a} + q_3 p_{3a} = 1$  with  $q_1, q_2, q_3$  constants) is adopted as a value unit. If there are nonbasics which are required for the production of the nonbasic that enters the value unit we can generalise the above observations. It would be found that:

i) Such of the nonbasics that enter directly or indirectly into their own production and enter, directly or indirectly, into the value unit would add to the number of possible solutions to the polynomial equation given by (4) above<sup>19</sup>.

<sup>18</sup> The price equation for the nonbasic at  $\lambda = 1/a_{33}$  gives in system A:

$$\frac{a_{13}}{a_{33}}p_{1a} + \frac{a_{23}}{a_{33}}p_{2a} + a_{03}w = 0.$$

With  $a_{33} > 0, a_{13}, a_{23} \ge 0$  this cannot be satisfied for positive prices. See also below p. 419.

<sup>19</sup> Thus consider two nonbasics in the above system (commodities 3 and 5) with the commodity 3 as a value unit and commodity 5 entering its production. We have the two systems differing in the method for commodity 1 as before. The price relations are given by (in system A):

$$(1 - a_{11} \lambda) p_{1a} - a_{21} \lambda p_{2a} - a_{01} W_a = 0$$
  
-  $a_{12} \lambda p_{1a} + (1 - a_{22} \lambda) p_{2a} - a_{02} W_a = 0$   
-  $a_{13} \lambda p_{1a} - a_{53} \lambda p_{5a} - a_{03} W_a = a_{33} \lambda - 1$   
-  $a_{14} \lambda p_{1a} + (1 - a_{55} \lambda) p_{5a} - a_{04} W_a = a_{35} \lambda$ 

ii) However the solutions that are added on are the rate(s) of profit equal to the rate of reproduction for the separate nonbasic or a group of interconnected nonbasics, as the case may be. These switch points however would have to be ruled out for the following reasons: If the rate of reproduction of a nonbasic (or a group of interconnected nonbasics) is smaller than the lower of the two maximum rates of profit for the two systems, the condition regarding the positivity of prices at those switch points will not be satisfied. In fact with the nonbasic as a value unit and a rate of profit equal to its rate of reproduction, one obtains, as *Sraffa* shows<sup>20</sup>, a 'formal' solution in which ''the price of every commodity is zero''. Thus the two production systems could 'formally' have a switch point which has to be ruled out since we consider only those switch points at which all prices are positive.

More importantly, such a value of the rate of profit at the switch point might well fall beyond the maximum rate of profit for at least one of the production systems. A fuller discussion on this issue appears in the Sraffa-Newman correspondence which is reproduced in the Appendix below. Sraffa argues there that instances of a nonbasic in the system having a rate of reproduction less than the maximum rate of profit for the system would be hardly met with and that the particular example of beans, a nonbasic of that type, which he employed in Appendix B of his book had to be invented in order to establish that, with such a nonbasic in the system, positivity of prices could not hold at a rate of profit equal to the rate of reproduction of that nonbasic.

Propositions similar to i) and ii) above can be proved in the case where the commodity with the alternative methods is a nonbasic, each of the other commodities common to the two systems having the same method. If there are other nonbasics in the system, which, while not entering either of the alternative methods of production, directly or indirectly, enter the value unit,

For system B coefficients in the first and second equations alone are different, the respective price equations being:

$$(1-b_{11} \lambda) p_{1b} - b_{41} \lambda p_{4b} - b_{01} W_b = 0$$
  
- b\_{14} \lambda p\_{1b} + (1-b\_{44} p\_{4b}) - b\_{04} W\_b = 0

The switch points for the two systems are obtained as before by solving for  $\lambda$  as in (4) above. The expression on the left hand side of (4) gives in this case a common factor, a polynomial in  $\lambda$  of degree at most two and at most two additional values for  $\lambda$  (i.e. two more than would have been obtained if only basics formed the value unit). This common factor is  $\{1-a_{33} \lambda\}(1-a_{55} \lambda)-a_{35} \lambda a_{53} \lambda\}$  which when equated to zero gives the value of  $r = \lambda - 1$ , equal to the rate of reproduction of the group of nonbasics. It will be noted that if  $a_{33} = 0$  and  $a_{35} = (0$  with  $a_{55} > 0$ , given) then  $\lambda = 1/a_{55}$ . This is the case when only commodity 5 of the two nonbasics requires itself in its own production. Similarly if  $a_{33} > 0$  but  $a_{35} = 0$  and  $a_{55} = 0$  then  $\lambda = 1/a_{33}$ . In both cases the solution for  $\lambda$  gives a rate of profit equal to the rate of reproduction of the separate nonbasics.

<sup>20</sup> Sraffa P., op. cit., Appendix B, p. 90-91.

directly or indirectly, then such nonbasics would not add to the maximum number of solutions for switch points between the two systems excepting in a formal way, as pointed out in ii) above.

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In the foregoing we have used the classification of commodities into basics and nonbasics to discuss the question of switching possibilities between two systems. Another classification which has been used more frequently in current discussions is that of decomposable and indecomposable systems. However, given a production system, the classification of the commodities involved into basics and nonbasics uses more of the available information about the system than does the classification of that system as decomposable or indecomposable. By stating that the system contains (or does not contain) nonbasics we would have already implied that the system is decomposable (or indecomposable). The classification of commodities into basics and nonbasics would further inform us as to which commodities in that system give rise to its decomposability. For, by their very nature, the basics in the system can be identified as forming a wholly interconnected group (we shall call this system, formed by all the basics in the system, the Basic system) while the nonbasics cannot do so since, while they require basics for their production, they are not themselves required in the production of the basics.

The additional information incorporated in the basic-nonbasic distinction is relevant to the discussion of switching possibilities between systems since it directly leads onto a distinction between two types of switches which have different consequences. A switch in the method for a basic implies that the two systems (each characterised by the method that it uses for that basic) would have different Basic systems, each with a maximum rate of profit, different from that of the other. On the other hand, a switch in the method for a nonbasic does not affect the maximum rate of profit of the system nor the prices of the basics, i.e. the Basic system is not affected in any way by a change in the method for a nonbasic. Another instance of the asymmetry between the two classes of commodities can be seen in this, that propositions concerning the switches in a basic (such as the maximum number of possible switches and the rate of profit at which a switch occurs) and the transition from one system to another that these switches imply can be derived from the consideration of the Basic system alone, ignoring the nonbasics, while such propositions concerning a nonbasic cannot be based on the consideration of nonbasics alone.

It would seem that some of the confusion which arose from a paper by D. Levhari<sup>21</sup> and which concerned the decomposability of a system and its relevance to the reswitching of techniques could have been avoided if the distinction between basics and nonbasics had been taken into account. Levhari in his paper claimed to have demonstrated that if there are n commodities in a system and if the  $i^{\text{th}}$  commodity (i = 1, 2, ..., n) had  $k_i$  alternative methods of production (so that there are  $\prod_{i=1}^{n} k_i$  possible systems of production) it is impossible that anyone system of production should switch back as the rate of profit continues to move in anyone direction. This claim was withdrawn later by Levhari and Samuelson<sup>22</sup>. They there explained that Levhari's original paper had accepted the possibility of a decomposable system's reswitching as 'established without question' by Ruth Cohen, Joan Robinson and P.Sraffa and it had attempted to show that such a reswitching could not happen in an indecomposable system. It should however be noted that Sraffa's demonstration of the possibility of reswitching was not limited to a decomposable case as Levhari had believed. When Sraffa takes the case of the alternative methods for a basic (having first considered briefly that of the alternative methods for a nonbasic) his argument does not require the existence of nonbasics in the system: the proposition concerning the possibility of reswitching of the basic holds whether the original system is decomposable or not. It is true that Sraffa makes a distinction between basic uses and nonbasic uses but this distinction is introduced only in order to facilitate comparison between the alternative methods within the same system at rates of profit at which the two methods are not equally profitable, i.e. away from the switch points. Each of the two commodities considered there (copper I and copper II) is basic to one or the other system. Off the switch point any comparison of the two alternative methods of producing copper at the prices of the system characterised by the use of copper I as basic implies treating the method producing copper II as a nonbasic in that system; the production matrix including both is decomposable<sup>23</sup>.

<sup>21</sup> Levhari D., A nonsubstitution theorem and switching of techniques, Quarterly Journal of Economics 1965, p.98-105.

<sup>22</sup> Levhari D. and Samuelson P., The nonreswitching theorem is false, Quarterly Journal of Economics 1966, p. 518-519. The Levhari Theorem was withdrawn when it was refuted conclusively by a number of writers (see Pasinetti L., Morishima M., Garegnani P., Bruno M., Burmeister E. and Sheshinski E.) in the Symposium on paradoxes in capital theory in the Quarterly Journal of Economics 1966, p. 504-583.

<sup>23</sup> Levhari's argument is based on the assumption that there are a number of alternative methods for producing each commodity and that each one of the possible systems of production consists of only basics. His statement on the nonreswitching of anyone system of production seems to have been suggested by the conjecture that when there is a wide range of known methods for each one of the several commodities the The case where two production systems have different basics, such as considered in Section I above, is a parallel instance where decomposability in the process of a comparison between the two systems arises. Each one of the two systems  $\mathcal{A}$  and  $\mathcal{B}$  appearing there is indecomposable and yet a comparison of the two systems implies the use of augmented systems  $\mathcal{A}^+$  and  $\mathcal{B}^+$  formed by adding to each system as nonbasics, the commodities which are basics only in the other. As we have seen such nonbasics peculiar to one or the other augmented systems of Section I have to be distinguished from the nonbasics common to *both* systems considered in Section II. In general, when the switching possibilities for basics, involving a transition from one system to another with a different Basic system, are being discussed nonbasics to both systems can be ignored<sup>24</sup>.

As an illustration of how a failure to specify whether the commodity that switches is a basic or nonbasic could be misleading, we may refer to Section III of the paper by *Bruno*, *Burmeister* and *Sheshinski* in the Symposium<sup>25</sup>. They consider there first the "canonical model" of *Samuelson*, with one capital good (which is basic) and one consumption good (which is nonbasic) in each production system. While the capital good is different in the two systems the consumption good is the same in both. The consumption good is numeraire and does not use itself in its own production. The authors state correctly, in this case, that there can be at most two switches. (There is only one commodity, the nonbasic consumption good, which is common to the two systems and the two production systems are characterised therefore by the nonbasic being produced by a different method in each. As the nonbasic does not use itself in either of these methods, the maximum number of switching points is only two.) After obtaining the sufficiency conditions for nonreswitching in this

probability that a number of methods should reswitch at the same point (i.e. the same system should return) would be very small. Levhari, however, claimed to have established rigorously the *impossibility* of such a reswitching and this claim was certainly wrong.

<sup>24</sup> We may conceive of a peculiar economic situation in which a nation consists of two or more separate economic communities having different customs and therefore, for instance, producing and consuming different kinds of food, etc. They are considered as forming a single statistical aggregate and therefore a single system. The production matrix for such a system would be 'completely decomposable'. (Mathematically, a square matrix A is called 'completely decomposable' when by identical arrangement of rows and columns it can be partitioned into  $\begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$  with  $A_{11}$  and  $A_{22}$ square.) In a completely decomposable system there are no basics as no commodity enters, directly or indirectly, into the production of all commodities in the system. Such a system could be subdivided into the independent economies which are combined to form that system and the commodities in each such economy classified as basics and nonbasics to that economy.

<sup>25</sup> Op. cit., p. 531-538.

simple case they point out the difficulty of generalising these conditions to cases involving more than one capital good in a single method of production. It is here that the error creeps in when they state (p. 536) "The latter fact [the difficulty of so generalising] can be seen by considering a case with one consumption good and two capital goods where the prices are clearly equations of the third degree. Thus in general there may be three switching points". In a production system with two basics and one nonbasic, with the nonbasic as numeraire, prices are not equations of the third degree in the rate of profit unless the nonbasic uses itself as means of production (a condition not present in the canonical model). Further we cannot conclude that the maximum number of switches between two production systems with two basics and one nonbasic in each would be in general three. If following the 'canonical model' we were to assume that the two capital goods (basics) in each system were, in all, four different basics and that the nonbasic did not enter its own production, the two systems have only one commodity, the nonbasic, common to them; there are at most four switching points. If the two capital goods in each system were the same two basics and the two systems were characterised by a different method for one of the basics the maximum number of switching points would be at most two. If the two systems had the same two basics, they differed only in the method for the common nonbasic and the nonbasic did not use itself in any of the alternative methods then the maximum number of switches would still be two. The maximum number of switching points would be three when the total number of *different* commodities entering, directly or indirectly, into at least one of the two alternative methods that switch is three<sup>26</sup>. No such condition is specified by the authors and it would seem that they arrived at the conclusion that the number of switches was in general three by counting the commodities in each system.

## IV

#### To sum up:

i) At a switch point the adjacent production systems differ in the method of production for only one of the commodities common to them. The maximum

<sup>26</sup> With two basics and one nonbasic in each system the maximum number of switches would be three when

i) the total number of distinct basics in the two production systems together is three and (a) the two systems are characterised by the use of a different method for a basic common to them, or, (b) the two systems differ only in the method for the nonbasic common to them and neither methods for the nonbasic uses the nonbasic. Or, alternatively when

ii) the total number of distinct basics in the two systems together is two; the two systems differ in the method for the nonbasic and the nonbasic uses itself in at least one of the methods by which it is produced. number of switching possibilities between two such systems is equal to the number of distinct (i.e. without double counting) commodities entering, directly or indirectly, into the two alternative methods which respectively characterise the two systems. Thus if it is a basic to both systems which has different methods in the two systems, the maximum number of switches would be equal to the total number of distinct basics in the two systems together; if it is a nonbasic which has different methods in the two systems, this maximum number is given by the total number of distinct basics in the two systems *plus* the number of distinct nonbasics entering, directly or indirectly, in at least one of the methods for that nonbasic.

ii) The choice of the value unit does not affect the maximum number of switching possibilities. Nonbasics which require themselves in their own production and which, while not entering, directly or indirectly, into the production of the commodity with alternative methods, do so enter the value unit, give additional formal solutions for switch points. These additional solutions would be ruled out for reasons given on p.417-8 above.

iii) The classification of commodities into basics and nonbasics in a given system uses more of the available information about the system than does the classification of the system as decomposable or indecomposable. The additional information incorporated in the former distinction is essential for the discussion of switching possibilities between two systems.

## Appendix

Professor Newman in his critique<sup>27</sup> of Piero Sraffa's "Production of Commodities by Means of Commodities" raised the issue concerning the necessary and sufficient conditions for all prices to be positive in a production system which includes nonbasics. These conditions (which he states in the article on p. 67) appear to him "to have little economic significance". His conclusion is that the presence of nonbasics in the system "will often not imply a positive price vector". This question of the economic interpretation of these conditions and the treatment of nonbasics were discussed in letters exchanged between Sraffa and Newman. I sought their permission, which they have kindly given, to publish the letters in full in this Appendix. I here summarise Newman's arguments as they appear on p. 66–67 of his article.

Newman first establishes that for a system containing only basics and in which 'labour consumes fixed levels of inputs irrespective of the rate of profit' there is always a solution giving a positive price vector and a positive rate of profit. He then considers a system which includes nonbasics. He gives

<sup>27</sup> Newman P., Production of Commodities by Means of Commodities, Schweizerische Zeitschrift für Volkswirtschaft 1962, p. 58–75.

a simple illustration of a system consisting of only two commodities, iron and corn, where iron (designated commodity 1) is nonbasic and corn (designated commodity 2) is basic. With  $p_1$  and  $p_2$  as prices of iron and corn respectively and r, the uniform rate of profit, the price equations in his example are:

$$\begin{array}{l} (1+r) \; 0.8 p_1 \,+\, (1+r) \; 0.3 p_2 \,=\, p_1 \\ (1+r) \; 0.2 p_2 \,=\, p_2 \\ 0.2 p_1 \,+\, 0.5 p_2 \,=\, 1 \end{array}$$

In this system if  $p_2 \neq 0$  then  $(1+r) = 1/a_{22} = 5$  and at that rate of profit  $p_1 = -5/4$  and  $p_2 = 5/2$ . Hence if  $p_2 \neq 0$  the solution contains a negative price. If  $p_2 = 0$ ,  $p_2 = 5$  and r = 1/4. He concludes that "in either case we have a contradiction of Sraffa's combined requirements that the system be in a selfreplacing state and that profit rate be uniform". Newman then states that the necessary and sufficient condition for such a production system having all positive prices is  $a_{11} < a_{22}$  (where  $a_{11}$  and  $a_{22}$  are the iron-iron and corn-corn coefficients respectively). The economic rationale of this condition seems obscure to him. He poses the choice that either we must abandon one of Sraffa's assumptions (that there is a uniform rate of profit and that the system is in selfreplacing state) or assume that nonbasics do not exist. He favours the course of 'abandoning the nonbasics'. He further adds that "this choice is reinforced by the consideration that the question whether a good is nonbasic is partly a matter of the degree of aggregation in the system". He concludes that "This result, that nonbasics will often not imply a positive price vector, means that the rather heavy emphasis placed on such commodities by Sraffa [he exemplifies them by luxury goods] seems misplaced".

The correspondence reproduced below centres on these issues.

Trinity College, Cambridge, England 4th June, 1962.

Dear Professor Newman,

Thank you for sending me your excellent article on my book. I have read it with great interest and I am sure that it will prove illuminating to many who have been puzzled by my work.

There are naturally some points of disagreement. Among these I shall refer only to your criticisms (p. 66-67) of my treatment of non-basic products. Have you not overlooked my Appendix B, to which the reader was referred to by a footnote on p. 28? It seems to say exactly the same thing as you say on p. 66. True, it says it in humdrum economic language, which is no doubt less elegant than mathematics. In this case, however, it has the advantage of making plain the economic circumstances which may give rise to a negative price for a non-basic, and which you find "obscure" (p. 67).

Besides, it makes it obvious how rare (if any) such cases must be in the real world. If, e.g., the ratio of net product to means of production (R) in a basic system is 25%, it will be pretty hard to find a single commodity (whether basic or not) which requires the using up of more than four units of itself in order to produce five units of it in a year. I certainly failed to discover any faintly realistic example of this which I could use, and had to invent those "beans".

When you say such instances occur "often" (p. 67) you must have been misled by your own example of a system consisting of a single basic and a single non-basic product – presumably concluding that  $a_{11} > a_{22}$  is no less probable than  $a_{11} < a_{22}$ . In a real system, however, there is not one but a large number of basic products, and the ratio R resulting from the *system* which they form is practically certain to be much smaller than the own ratio of anyone *separate* non-basic (or any of such small groups of interconnected non-basics as may exist).

You find a further ground for attacking the distinction between basics and non-basics in the supposition of its being "partly a matter of the degree of aggregation in the system" (p. 67). Now aggregation is the act of the observer, whilst the distinction is based on a difference in objective properties. I have argued, for instance, that a tax on the price of basics will lower the general rate of profits for a given wage, whereas a similar tax on non-basics will leave the rate of profits unchanged. Surely, to answer this, one must prove the alleged consequence does not follow, instead of drowning the distinction through an appropriate degree of aggregation.

Thank you again for your article. If I may hope for more, it is that you will not really leave your reader to shift for himself in the maze of multiple-product industries.

> Yours sincerely, P.Sraffa

Department of Economics, The University of Michigan, Ann Arbor

June 8, 1962.

Dear Mr. Sraffa,

426

Thank you so much for your letter, and for your kind words concerning my article. It was a relief to learn that I had not badly misinterpreted your ideas, as I feared I might have done.

I can come half-way to meet your criticisms of my treatment of non-basics. I now think that there is some economic meaning to Gantmacher's conditions (p.92) for the positivity of prices. Let us designate the reducible system Ap = cp by

$$\begin{bmatrix} \mathcal{A}_B & 0 \\ \mathcal{A}_{BN} & \mathcal{A}_N \end{bmatrix} \begin{bmatrix} \mathcal{P}_B \\ \mathcal{P}_N \end{bmatrix} = c \begin{bmatrix} \mathcal{P}_B \\ \mathcal{P}_N \end{bmatrix}$$

where  $\mathcal{A}_B$  and  $\mathcal{A}_N$  are the square 'internal' coefficient matrices for basic and non-basic goods respectively,  $\mathcal{A}_{BN}$  is the (in general nonsquare) matrix of coefficients of basic goods used in non-basic good manufacture,  $P_B$  and  $P_N$  are subvectors of the respective prices, and c is  $\mathcal{A}$ 's dominant latent root. Then we can consider  $\mathcal{A}_B$  and  $\mathcal{A}_N$  as themselves matrices like  $\mathcal{A}$ , with dominant latent roots  $c_B$  and  $c_N$  respectively, and associated eq. 'rates of profit'  $r_B$  and  $r_N$ .

Then Gantmacher's necessary and sufficient condition for positivity of  $P_B$  and  $P_N$  may be expressed as  $r_B < r_N$ , i.e., the rate of profit in the basic system must be strictly less than the rate of profit of the 'internal' non-basic system. This seems to have economic meaning, though I am not sure about its significance. I confess that it does not seem to me to be obvious that we will usually have  $r_B < r_N$ , but I am open to argument. It seems to me that more empirical considerations would have to be brought in.

I would not have brought in the point about aggregation if I had not already made the earlier, and I think stronger, point. I do wonder a little about your mention of 'objective properties'. All we ever have is what we observe, or more strictly, what we classify. I personally find it difficult to think in terms of industries when considering production, and think more naturally of processes. For this reason, I think further discussion of this point would not be useful, since I imagine that we would both agree that the Part II analysis of processes is a considerable step forward. I have not thought about the role of aggregation in the latter context.

Your invitation to work on Part II of the book is very enticing. My free time is rather limited just now, and I suspect it will take much harder work than Part I. But I might steal time to work at it.

With best wishes.

Yours sincerely, Peter Newman

Trinity College, Cambridge 19th June, 1962.

Dear Professor Newman,

Thank you so much for your letter.

I am, of course, delighted, and grateful, that you can come half-way to meet me on the subject of non-basics, and I only regret to be unable to move the other half: I cannot yield an inch on this point! You speak of a non-basic system and proceed to compare it with the basic system: I say that there is no such thing as a non-basic system. You also refer to "the rate of profit of the internal non-basic system": again, I say there is no such thing.

It is in the nature (or, if you wish, the definition) of basic goods to be interconnected and form a system. It is, on the other hand, the peculiarity of nonbasics to be unconnected with one another, and they are incapable of forming an independent system. At best, each of them can be formally treated as constituting a separate single-commodity system, with its own rate of profits: this rate (for each separate non-basic) can be compared with the rate of the basic system. It is *a priori* extremely unlikely that any individual rate will be smaller than that of the basic system, composed, as the latter is, of many products, all used directly or indirectly in one another's production. It has not been possible to find a reasonable case in reality in which the rate is smaller (and this is not a minute, hidden property that requires elaborate investigation for spotting it).

If I may go over the ground again. The immense majority of non-basics are not used in production, not even in their own production: so they do not even form individual systems. Some (mainly animals and plants) are used each in its own reproduction, and form individual systems. A few may be linked with one or two others, because of mixing, or cross-breeding, or if the length of gestation brings out the egg-hen dicotomy. And that is all.

The third class, which is the least numerous and may just be worth mentioning for the sake of completeness, is the source of all the trouble.

With many good wishes.

428

Yours sincerely, P.Sraffa

#### Summary

#### On the Maximum Number of Switches Between Two Production Systems

This paper discusses, adopting Piero Sraffa's classification of commodities into 'basics' and 'nonbasics', the question of the maximum number of switches between two production systems in the most general case where the two systems are characterised by 'basics' not all common between them and where they may include 'nonbasics'. This maximum number is given by the number of distinct (i. e. without double counting) commodities entering, directly or indirectly, into the two alternative methods that characterise the two systems – the two systems adjacent at a switch point differing in the production method for only one of the commodities common between them. The paper brings out the particular advantage of the 'basic-nonbasic' distinction for this discussion. In the Appendix are published letters exchanged between Piero Sraffa and Peter Newman on the role of 'nonbasics'.

#### Zusammenfassung

#### Über die maximale Anzahl von Wechseln zwischen zwei Produktionssystemen

Dieser Aufsatz diskutiert, ausgehend von Piero Sraffas Klassifikation der Güter in « basics » und « nonbasics », die Frage der maximalen Anzahl der Wechsel zwischen zwei Produktionssystemen im allgemeinsten Fall. Hier zeichnen sich diese Systeme durch verschiedene « basics » aus, wobei aber beide Systeme « nonbasics » enthalten mögen. Diese maximale Anzahl wird durch die Zahl der verschiedenen Güter (d. h. Doppelzählungen sind ausgeschlossen) bestimmt, welche direkt oder indirekt in die zwei alternativen Methoden eingehen, welche die beiden Produktionssysteme charakterisieren. Dabei besitzen beide Systeme an einem Wechselpunkt eine grosse Ähnlichkeit, da sie nur einen Unterschied in der Produktionsmethode bezüglich eines der gemeinsamen Güter aufweisen.

Der Aufsatz zeigt den besonderen Vorteil der Unterscheidung in «basics» und nonbasics» für diese Diskussion. Im Anhang wird der Briefwechsel zwischen Piero Sraffa und Peter Newman über die Rolle der «nonbasics» publiziert.

## Résumé

#### Le nombre maximum de changements entre deux systèmes de production

En adoptant la classification de Piero Sraffa des biens en « basics » et « nonbasics », l'article discute la question du nombre maximum de changements entre deux systèmes de production dans le cas le plus général où les deux systèmes sont caractérisés par des « basics » étant entre eux tout à fait différents et où les systèmes pourraient comprendre des « nonbasics ». Le nombre maximum est déterminé par la quantité de biens différents (c'est-à-dire sans dénombrement double) entrant directement ou indirectement dans les deux méthodes alternatives qui caractérisent les deux systèmes. Les deux systèmes, similaires à un point de changement, ne différent dans la méthode de production qu'en un des biens communs entre eux. L'article montre l'avantage particulier de la distinction entre « basics » et « nonbasics ». Dans l'appendice, on publie des lettres échangées entre Piero Sraffa et Peter Newman sur le rôle des « nonbasics ».