I. Introduction

§ 1.1 General Framework and Plan of the Paper

Until a few years ago one of the main assumptions of the neoclassical theory was the knowledge, more or less absolute, of future economic events. This deterministic approach implies that all individuals are able to forecast with certainty the future value of the main variables of the model (for instance the interest rate earned on accumulated savings in a life-cycle model). The assumption of perfect foresight of such important variables is surely a restrictive hypothesis.

The effect of uncertainty and risk-aversion upon rational decision rules for individuals (as well as institutions) to use in selecting optimal consumption and saving patterns and optimal portfolios have increasingly engaged the attention of theoretical and professional economists of the capital markets and of business finance.

The purpose of this paper is to provide a simple framework in which we explore the consequences of the introduction of interest uncertainty on the consumption- and saving-plans of an individual first and on the optimal accumulation rate of an entire dynasty or class later. In other words we try to answer the question of what is going to happen when uncertainty is introduced into a deterministic life-cycle model.

Our analysis will show that while the variance of the uncertain rate of return on capital has a definite effect on the optimal variables of the model, the risk-
aversion measure has an ambiguous effect on them. For instance the consumption rate, as the risk-aversion increases, is initially positively affected while later it tends to decrease.

In the final part of the analysis, possibly the most original one, we consider a particular property of the model: once the rate of growth of the mean wealth of an entire class or group of people (to which we attach a specific risk-aversion) has been determined, it is possible to study the accumulation of the financial capital from an historical point of view. More precisely, assuming that the risk-aversion is a function of the size of the financial wealth held, it will be shown that as the generations pass the economy converges, under some assumptions, towards a two-class society (where one class has a high capital per capita and the other a lower one).

**Plan of the paper.** The possible effects of interest uncertainty and risk-aversion on the consumption and saving rates will be analysed in the following § 1.2. In Section II we study the various effects of interest uncertainty in a life-cycle model with a risky capital market. The basic assumptions are specified in § 2.1, followed in § 2.2 by an analysis of the consumption rate under uncertainty and in § 2.3 by an analysis of the path of accumulation of capital under uncertainty. In the following Paragraph some numerical results for different combinations of the parameters will be given. In Section III we focus on a particular outcome of the analysis which seems to confirm, within certain limits, the classical assumption of a two-class society, but with a completely different perspective. A Conclusion will bring the paper to a close.

§ 1.2 A Possible Effect of Interest Uncertainty on Consumption and Accumulation Rates

In this paper uncertainty is reflected uniquely in the expected interest rate by attaching to it a positive variance. The introduction of interest uncertainty into a life-cycle model where individuals make their plans in order to maximize the expected value of the flow of the discounted utilities from consumption, may have the following (distinct but balancing) general effects.

First of all it may be argued that uncertainty may increase the propensity to save of an individual in order to create a special reserve of assets in the eventuality that the more pessimistic previsions could come true. But on the other hand it may be stressed that uncertainty may reduce the propensity to save of an individual since he may fear that he will not be able to benefit of the services of his accumulated savings. And since these two effects play in opposite directions, in a fairly general model they would offset each other at some point, where an optimum would be obtained.
The analysis of a general life-cycle model with interest uncertainty is very difficult without making various simplifying assumptions. Moreover, as Champernowne (1969, p.42) claims "... the main difficulties in discovering the effects of risk on the inducement to save are those of choosing sensible simplifying assumptions. Apparently slight variations in the choice of assumptions may lead to striking modifications of the conclusions".

But what are the consequences of a positive variance on the optimal consumption and accumulation rates? In this respect Champernowne argues that when there is a positive variance in the expected rate of interest and if we consider a model where an individual starts with an initial positive bequest (from which he will derive all his income) then the effect of the variance is to discourage initial consumption and stimulate initial savings. Our analysis will show that this conclusion is not always general as expected.

II. The effects of Interest Uncertainty in a Simple Life-Cycle Model

§ 2.1 The Basic Assumptions

As specified in the Introduction the purpose of this paper is to provide a simple theoretical framework in which the various effects of uncertainty and risk-aversion may be analysed. This will be done by assuming a unique risky capital market and disregarding labour income (the latter is a common assumption in the literature). In addition we make the following assumptions:

Time horizon. If uncertainty about the rate of return on capital is introduced into a life-cycle model where individuals make their plans with a finite horizon, the problem is not easy to solve from an analytical point of view. Therefore we assume an infinite-time horizon. This assumption is equivalent to considering a dynasty of wealth owners and is less restrictive that at first sight it could seem, since some of the results that we shall obtain do not diverge from those obtained by Merton (1969 & 1971) in the context of a finite time horizon.

Utility function. We assume an additively separable utility function yielding constant relative risk-aversion (i.e. an iso-elastic marginal utility) of the form

$$ U(c_t) = \frac{1}{2} (c_t)^a, $$

where $c_t$ is the consumption rate at time $t$. While most of the results of our analysis are valid for the more general case $a < 1$, we shall confine our analysis to the case $a < 0$. In this case the Arrow-Pratt relative risk-aversion measure will be denoted as $R_R = 1 - a$.

3 In a similar example, where there is no initial positive bequest, but where consumption is derived from a flow of future uncertain incomes, Champernowne (1969) arrives at an opposite conclusion.

4 More precisely the basic problem is: Max $\int_0^T U(c_t) dt$, with the first order condition $U'(c_0) = E(e^{rt} U'(c_t))$. But since $U'(c_t) = f(r, \ldots)$ the expression $E(e^{rt} f(r, \ldots))$ is not easy to solve.
§ 2.2 The Optimal Consumption Rate

Under these assumptions an individual who, at time $t_0$, inherits a financial capital stock $B_0$, from which he will derive all his income, will make his consumption plans in order to maximize, in continuous time, the flow of the discounted utilities from consumption. Since utility is isoelastic and the investment opportunity itself is stationary, the stationarity of the whole problem ensures the optimality of the policy of consuming, at any time, at a rate proportional to total wealth (i.e., $c_t = cB_t$). Hence we may write:

$$\Phi(B_t) = \max \int_0^\infty e^{-\delta t} U(cB_t) \, dt,$$

(1)

where $c$ is the constant (fixed) proportion of capital consumed;
$\delta$ is the pure time-preference (or subjective discount rate);
$B_t$ is the total financial capital stock at time $t$.

It is at this point that uncertainty may be introduced. Let us assume there is a single capital market with a serially independent random return $R(t)$ with positive mean $\bar{r}t$ and finite variance $\sigma^2$ proportional to the period of holding $\theta$ (note that higher moments disappear in the limit since they are less than proportional to time $\theta^5$); Wiener processes and Brownian motions conform to these requirements. Therefore we can define $^6$:

$$E(R(t)) = \bar{r}(t),$$

the expected rate of return on the (risky) assets,

where $\text{Var } (R(t)) = \sigma^2(t) > 0$. (2)

From the definition of the utility function we can rewrite relation (1) as follows $^7$:

$$\Phi(B_t) = \max \int_0^\infty \frac{1}{a} B_0^a \int_0^\infty e^{a[r-(1-a)^2/c-\delta]t} \, dt,$$

(3)

$^5$ The property of random return (which greatly simplifies the analysis) is basically the random-walk hypothesis considered in Cootner (1964).

$^6$ In a continuous time model this corresponds to assuming a lognormally distributed random variable of the form:

$$Y = e^{\tilde{p}}$$

where $\tilde{p} = N(r - \frac{\sigma^2}{2}, \sigma^2)$.

$^7$ The result given in relation (3) can be obtained considering the following discrete-time budget
Integrating (3) we obtain the following result:

$$\Phi(B_t) = \max \left\{ \frac{c^a}{a} B_0^a \left( a \left[ r - (1 - a) \frac{V^2}{2} - c \right] - \delta \right)^{-1} \right\},$$

(4)

differential equation which is the generalisation of the continuous-time budget equation under uncertainty (I owe this point to Professor Mirrlees):

$$\frac{d}{dt} B = B d R - c B d t. \quad (a)$$

Therefore, as we want to find the value of $\frac{1}{a} B^a$, we write:

$$d \left( \frac{1}{a} B^a \right) = B^{a-1} d B + \frac{1}{2} (a - 1) B^{a-2} (d B)^2 + \ldots =$$

$$= B^{a-1} (B d R - c B d t) + \frac{1}{2} (a - 1) B^{a-2} B^2 \nu^2 d t =$$

$$= B^a \left( \left[ (a - 1) \frac{1}{2} \nu^2 - c \right] dt + d R \right). \quad (b)$$

Hence:

$$\frac{d}{dt} E \left( \frac{1}{a} B^a \right) = \left[ (a - 1) \frac{\nu^2}{2} - c \right] E(B^a) + \lim_{\theta \to 0} \frac{1}{\theta} E(R(t + \theta) - R(t)) \quad (c)$$

and since:

$$\lim_{\theta \to 0} \frac{1}{\theta} E(R(t + \theta) - R(t)) = E(R) = r, \quad (d)$$

we get

$$\frac{d}{dt} E \left( \frac{1}{a} B^a \right) = E(B^a) \left[ r - (1 - a) \frac{\nu^2}{2} - c \right]. \quad (e)$$

Hence

$$E \left( \frac{1}{a} B^a \right) = \frac{1}{a} B_0^a e^{r \nu^2 t - c} \left( 1 - e^{(1 - a) \frac{\nu^2}{2} + c} \right), \quad (f)$$

so that:

$$c^a \int_0^\infty e^{-\delta t} E \left( \frac{1}{a} B^a \right) dt =$$

$$= \frac{1}{a} c^a B_0^a \int_0^\infty e \left( a \left[ r - (1 - a) \frac{\nu^2}{2} - c \right] - \delta \right) dt. \quad (g)$$

Provided that:

$$a \left[ r - (1 - a) \frac{\nu^2}{2} - c \right] - \delta < 0, \quad (6)$$

(which is reconsidered in relation (7) below); otherwise for negative values of $a$ we would obtain:

$$\Phi(B_t) = \max(-\infty).$$
and hence differentiating it partially with respect to \( c \) we obtain the following first order condition for a maximum, where a star denotes optimal values 9, 10:

\[
c^* = \frac{\delta - ar}{1-a} + \frac{v}{2}.
\]  

Relation (5) gives us the constant consumption rate that our individual will choose under rationality. The first conclusion to be drawn is that \( c^* \) is independent of time and of the initial bequest \( B_0 \). This latter result is due to the fact that here the relative risk-aversion is independent of wealth.

At least from an analytical point of view \( c^* \) may be negative. Solving it with respect to \( v \) (the variance of the interest rate) \( c^* \) turns out to be negative as long as \( v > \frac{2ar - \delta}{a(1-a)} \). Hence in this model the optimal consumption rate would be negative (i.e. zero) only for very high values of the variance (or for combinations of very high values of the variance and the risk-aversion). For economically reasonable values of the parameters as \( \delta = 0.03, r = 0.06 \) and \((1-a) = R_R(B) = 3\) we would have a negative (i.e. zero) consumption rate for values of the variance higher than 0.05, an extremely high value indeed.

With respect to the risk-aversion measure the optimal consumption rate equals the rate of utility discount (i.e. \( c^* = 0 \)) when \((1-a) = R_R(B) = 1\), while it tends to zero as \((1-a) = R_R(B) \rightarrow \infty\) as shown in Figure 1.

Substituting now the result obtained in (5) into (6) above we obtain the inequality \( \frac{ar - \delta}{1-a} < 0 \), which is always true since we have assumed negative values of \( a \) (and positive of \( r \) and \( \delta \)). Hence this result confirms the validity of relation (5).

At this stage it may be worth studying the influence of the parameters \( r, v, \delta, a \) on the consumption rate; this can be done by differentiating it partially with respect to each individual parameter:

\[
\begin{align*}
\frac{\partial c^*}{\partial r} &= \frac{a}{1-a} > 0 \\
\frac{\partial c^*}{\partial v} &= \frac{a}{2} < 0 \\
\frac{\partial c^*}{\partial \delta} &= -\frac{1}{1-a} > 0 \\
\frac{\partial c^*}{\partial a} &= (1-a)^2 \left( \frac{(1-a)^2}{2} \cdot \frac{v}{2} - r + \delta \right).
\end{align*}
\] 

9 The second order condition is also satisfied.

10 Note that if consumption is a constant proportion of wealth and (indirectly) of the mean rate of return on wealth we can measure the utility with respect to consumption, wealth and income at the same time. This procedure is common to many optimal savings models, such as the life-cycle savings hypothesis of Brumberg/Modigliani/Ando that we are considering here and the permanent income hypothesis of Friedman.
Hence an increase in the mean rate of interest \( r \) and in the subjective discount rate \( \delta \) has a positive effect on consumption. Both results are scarcely surprising. The former has been found in most stochastic models; the latter confirms that an increase in pure impatience stimulates, normally, initial consumption. Or, the other way round, given the market interest rate, a rise in \( \delta \) lowers the demand for wealth. Indeed a high subjective rate of discount encourages high consumption by young people, financed by borrowing against future incomes, so that the total demand for financial wealth turns out to be negative.

Relation (8) shows that an increase in the variance of the interest rate has a negative effect on consumption; hence uncertainty raises the intensity of capital, since lower consumption rates imply a higher capital. This aspect will be reconsidered below when analysing the accumulation rate.

Relation (10) is more ambiguous and needs further investigation. The more an individual is risk-averter, the smaller \( a \) will be. Solving the right-hand side of (10) with respect to \( a \) we get:

\[
\frac{\partial c^*}{\partial a} < 0 \quad \text{for} \quad 1 - \left(\frac{2(r - \delta)}{v}\right)^{1/2} < a < 0
\]

\[
\frac{\partial c^*}{\partial a} > 0 \quad \text{for} \quad a < 1 - \left(\frac{2(r - \delta)}{v}\right)^{1/2},
\]

with \( c^*(\text{max}) \) for \( a = 1 - \left(\frac{2(r - \delta)}{v}\right)^{1/2} \).

Since \( 1 - \left(\frac{2(r - \delta)}{v}\right)^{1/2} \) should always be negative\(^{11} \), the first conclusion that we can draw is that consumption will be maximum for \( a = 1 - \left(\frac{2(r - \delta)}{v}\right)^{1/2} \) and not for \( a = 0 \) as we could have reasonably expected. This corresponds to saying that in the interval \([1 - \left(\frac{2(r - \delta)}{v}\right)^{1/2}, 0]\) the more risk-averse an individual is the more he will consume. On the contrary in the interval \([-\infty, 1 - \left(\frac{2(r - \delta)}{v}\right)^{1/2}\] \) the opposite applies: more risk-aversion means less consumption.

\[\text{§ 2.3 The Optimal Path of Accumulation Under Uncertainty}\]

If uncertainty is introduced into an economic growth model, the model becomes a stochastic process, with the variables (defining the state of the economy) clearly becoming random variables whose probability varies with time. In this

\(^{11}\) Note that \( 1 - \left(\frac{2(r - \delta)}{v}\right)^{1/2} = \left(\frac{v - \left[\frac{v^2}{2} + 2v\left(r - \delta - \frac{v}{2}\right)\right]^{1/2}}{v}\right) \) should always be negative since in a life-cycle model the expression \( r - \delta - \frac{v}{2} \) must be positive to make economic sense; as only a (mean) interest rate greater than the discount rate assures a positive accumulation of capital in a life-cycle model. Should the contrary apply, then \( \frac{\partial c^*}{\partial a} > 0. \)
model we have assumed that the uncertainty process is such that the mean and variance are proportional to time. And since the variance of the rate of return on the financial capital stock is positive, it only makes sense to discuss the expected (or average, or mean) value of future wealth of a dynasty or class, since, as said, it is a function of a random variable.

To do this we follow the consumption plan obtained above and we write the rate of growth of the mean wealth of a particular dynasty or class:

\[ g^* = r - c^* \]

\[ = r - \left( \frac{\delta - ar}{1-a} + a \frac{v}{2} \right) \]

\[ = \frac{r - \delta}{1-a} - a \frac{v}{2}, \]  

which is always positive since \( r \) is always greater than the subjective rate of discount (\( \delta \)) to make economic sense, and since we are considering negative values of a only. Therefore an equal investment in portfolios having the same average rate of returns over many years will not yield identical values of terminal wealth if their variances differ.

Again we note that \( g^* \) is independent of time and of the size of the initial bequest. On the other hand it depends on all parameters of the model. To measure their influence we differentiate \( g^* \) partially with respect to the single parameters:

\[ \frac{\partial g^*}{\partial r} = \frac{1}{1-a} > 0 \]  

\[ \frac{\partial g^*}{\partial v} = -\frac{a}{2} > 0 \]  

\[ \frac{\partial g^*}{\partial \delta} = \frac{-1}{1-a} < 0 \]  

\[ \frac{\partial g^*}{\partial a} = (1-a)^2 (r - \delta - \frac{v}{2}(1-a)^2) = -\frac{\partial c^*}{\partial a}. \]

Hence we note that an increase in \( r \), the mean rate of return, and in \( v \), the variance, have both a positive effect on the rate of growth of the mean wealth. On the other hand an increase in the subjective discount rate \( \delta \) has a negative effect on \( g^* \).

Again, as already pointed out by studying the consumption rate, we note that an increase in the variance of the risky asset has a positive effect on the mean accumulation of wealth so that the general proposition that "risk and uncertainty raise the intensity of capital" is in this context confirmed.

Relations (16) and (17) are not surprising and could have been anticipated from relations (8) and (9) related to \( c^* \). Note, moreover, that relation (15) in particular
must be interpreted as a micro-relation; as a matter of fact it does not take account of the fact that in a neoclassical model as the aggregate capital intensity rises, \( r = f_K \) tends to fall.

Let us now turn to relation (18) which is, as relation (10), ambiguous in sign. As shown in (18) the value of \( \partial g^*/\partial a \) is the opposite of the value of \( \partial c^*/\partial a \). Hence we can write (for an appreciation of the value \( 1 - (2(r - \delta)/v)^{1/2} \) cf. Footnote 11, page 413):

\[
\begin{align*}
\partial g^*/\partial a > 0 & \quad \text{for } 1 - (2(r - \delta)/v)^{1/2} < a < 0 & (19) \\
\partial g^*/\partial a < 0 & \quad \text{for } a < 1 - (2(r - \delta)/v)^{1/2} & (20) \\
\text{with } g^*(\text{min}) & \quad \text{for } a = 1 - (2(r - \delta)/v)^{1/2}. & (21)
\end{align*}
\]

This means that in the interval \([1 - (2(r - \delta)/v)^{1/2}, 0]\) as \( a \) increases (which means that our subject becomes less and less risk averse) \( g^* \) will tend to decrease monotonically, however never reaching zero. (Note that \( g^* \) would become zero for \( a = 0 \) and \( r = \delta \): this would be a limit case in a life-cycle model.) On the other hand for values of \( a \) (the elasticity of the utility function) lower than the value \( 1 - (2(r - \delta)/v)^{1/2} \) the above derivative is negative: this means that a decrease in the relative risk-aversion will lead to a higher capital accumulation.

§ 2.4 Some Numerical Results

The value of \( c^* \) and \( g^* \), the optimal consumption and accumulation rates, for several combinations of the parameters are shown in two Tables below. Since we are considering a stochastic model we shall focus on the effect of risk-aversion and of the variance on the two important variables of the model.

The values of the parameters are econometrically reasonable for mixed economies like Western Europe, U.K. or the U.S. The values for \( r \), the mean rate of return on capital and of its variance are also representative of an average risky portfolio.

In Table 1 we show the behaviour of optimal consumption and mean accumulation when the risk-aversion changes and the other parameters are frozen. As anticipated the consumption rate \( c^* \), as the risk-aversion increases, first grows, attains a maximum at \((1 - a)^+ = 3.464\) and then slowly decreases reaching zero for \((1 - a) = R_R(B) = 24.51\), a very high value for the risk-aversion indeed. As it may be deduced from the figures, the sensitivity of \( c^* \) and \( g^* \) to a change in risk-aversion decreases as risk-aversion increases.

From Table 2 we deduce the behaviour of consumption and mean accumulation when the variance of \( r \) changes: an increase in the variance causes a (linear) decrease in the consumption rate and a (linear) increase in the rate of growth of mean wealth.
Table 1
Optimal Consumption (c*) & Accumulation (g*) for Different Combinations of the Risk-Aversion Measure (with one risky asset only)
When \( \delta = 3\% \), \( r = 6\% \) and \( v = 0.005 \) (All variables in percentages)\(^{12}\)

<table>
<thead>
<tr>
<th>((1 - a) = R_R(B))</th>
<th>(c^*)</th>
<th>(g^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.00</td>
<td>3.00</td>
</tr>
<tr>
<td>2</td>
<td>4.25</td>
<td>1.75</td>
</tr>
<tr>
<td>3</td>
<td>4.50</td>
<td>1.50</td>
</tr>
<tr>
<td>3.464(^{13})</td>
<td>4.52 (max)</td>
<td>1.48 (min)</td>
</tr>
<tr>
<td>4</td>
<td>4.50</td>
<td>1.50</td>
</tr>
<tr>
<td>5</td>
<td>4.40</td>
<td>1.60</td>
</tr>
</tbody>
</table>

with: \( a \) the elasticity of the u-function \[ U(c_t) = \frac{1}{a}(c_t)^a \];

\((1 - a) = R_R(B)\) the Arrow-Pratt relative risk-aversion measure;
\( \delta \) the subjective discount rate;
\( r \) the mean rate of return on capital;
\( v \) the variance of \( r \);
\( c^* \) the optimal consumption rate (see relation (5));
\( g^* \) the optimal rate of growth of the mean wealth (see relation (14.b));

Table 2
Optimal Consumption (c*) and Accumulation (g*) Rates for Different Combinations of the Variance
(with one risky asset only)\(^{14}\) for \( \delta = 3\% \), \( r = 6\% \) and \( (1 - a) = R_R(B) = 3 \)
(All variables in percentages)

<table>
<thead>
<tr>
<th>(v)</th>
<th>((1 - a)^{15})</th>
<th>(c^*)</th>
<th>(g^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>7.75</td>
<td>4.90</td>
<td>1.10</td>
</tr>
<tr>
<td>0.003</td>
<td>4.47</td>
<td>4.70</td>
<td>1.30</td>
</tr>
<tr>
<td>0.005</td>
<td>3.46</td>
<td>4.50</td>
<td>1.50</td>
</tr>
<tr>
<td>0.007</td>
<td>2.93</td>
<td>4.30</td>
<td>1.70</td>
</tr>
<tr>
<td>0.009</td>
<td>2.58</td>
<td>4.10</td>
<td>1.90</td>
</tr>
<tr>
<td>0.010</td>
<td>2.45</td>
<td>4.00</td>
<td>2.00</td>
</tr>
<tr>
<td>0.011</td>
<td>2.34</td>
<td>3.90</td>
<td>2.10</td>
</tr>
<tr>
<td>0.015</td>
<td>2.00</td>
<td>3.50</td>
<td>2.50</td>
</tr>
</tbody>
</table>

\(^{12}\) Note that \( c^* + g^* = r \).
\(^{13}\) In this example \( a^* = 1 - (2(r - \delta)/v)^{1/2} = -2.464 \) is the value for which \( \partial c^*/\partial a = \partial g^*/\partial a = 0 \), i.e. where \( c^*(\text{max}) \) and \( g^*(\text{min}) \).
\(^{14}\) The list of symbols is given in Table 1 above.
\(^{15}\) \((1 - a)^{15} = (2(r - \delta)/v)^{1/2} \) is the value of the risk-aversion for which \( \partial c^*/\partial a = \partial g^*/\partial a = 0 \), i.e. \( c^*(\text{max}) \) and \( g^*(\text{min}) \).
Figure 1
The Behaviour of Optimal Consumption ($c^*$) and Optimal Accumulation ($g^*$) as the Risk-Aversion ($1 - a$) and Variance change
(for $\delta = 0.03$, $r = 0.06$)

Notes:
1. The list of symbols is given in Table 1 above.
2. For $(1 - a) = R_R(B) = \text{relative risk-aversion} = 1$, we obtain always $c^* = \delta$ and $g^* = r - \delta$.

Figure 1 shows graphically the behaviour of the two variables $c^*$ and $g^*$ as the relative risk-aversion and the variance vary. It can easily be seen that when $1 - a = R_R(B) = (2(r - \delta)/v)^{1/2}$ the consumption rate reaches a maximum, while at the same abscissa $g^*$, the mean accumulation rate, is minimum.

Two notes are in order here. First of all it may be worth stressing that after reaching a maximum the consumption rate slowly decreases and reaches zero for very high values of the risk-aversion measure (the greater the smaller the variance is). Exactly the opposite applies for the mean accumulation rate: after reaching a minimum it grows slowly, but steadily, towards $r$, reaching it for very high values of the risk-aversion measure $1 - a = R_R(B)$.

Secondly it may be worth considering the implications of these findings. If we consider the value $(1 - a)^+ = R_R(B)^+ = (2(r - \delta)/v)^{1/2}$ at which $\partial c^*/\partial a = \partial g^*/\partial a = 0$, yielding a maximum for optimal consumption and a minimum for the rate of growth of mean wealth, we see that for econometrically reasonable values of the parameters the value of $(1 - a)^+$ turns out to be around $2.5 - 4$. Compar-
ing this result with the estimates of the average relative risk-aversion of Flemming (1974) and Bevan (1974) which hover around 3, with a range 2—4.5, we may deduce that in this model the behaviour of the average consumer would be to maximize consumption (and hence minimize g*). The intuitive reason for there being a maximizing value of (1 — a) = R_R(B) may be seen in the counteracting effects of risk-preference and intertemporal preference.

III. Back to the Classical Assumption of a Two-Class Society?

In the analysis which follows we assume that the relative risk-aversion measure \( R_R(B) = - B \cdot U''(B)/U'(B) = 1 - a \) is a decreasing function of the financial capital stock B held by an individual. This assumption does not necessarily contradict Arrow’s (1965) original argument (which has been criticized by Stiglitz (1968, p. 742)) following which the relative risk-aversion is an increasing function of the total capital stock. As a matter of fact it may be argued, inter alia, that the individuals with low financial capital stock might have a more than average human capital stock, which is a part of the total capital stock held by an individual\(^{16}\).

Armed with this definition we may argue that, from an historical point of view, the results obtained in the preceding Section II show that the classes which have accumulated more wealth (hence at a fast rate) are not exclusively those less risk-averse; as a matter of fact it can be seen that those which have been very high risk-averse (and hence with low financial capitals) would have accumulated at a fast rate as well, at least from a relative point of view. Besides, those who are in the middle classes (and hence are middle-risk-averse) will consume more and hence accumulate less.

Let us consider the dynamics of this process, by focusing on the behaviour of a large dynasty (or even a sub-class) since it makes sense to talk of the rate of growth of mean wealth of a rather large number of people only. When an individual (member or founder of a dynasty) makes, at time \( t_0 \), his consumption plans he has a specific risk-aversion (or elasticity of marginal utility) which, according to our assumption above, is a function of the size of his wealth. This risk-aversion measure will be, basically, the same for the cohort (dynasty or sub-class) to which he belongs and should always remain the same since we are considering a constant elasticity utility function.

However it is not unnatural to assume that the following generation will revise the plans according to the new situation. In this way the size of the bequest received from the preceding generation will determine, according to our assumption, the relative risk-aversion and hence the new optimal plan.

\(^{16}\) For a more exhaustive treatment of the argument cf. Baranzini (1976, pp. 139-42).
This process, under certain assumptions which we specify in the next pages, will continue through time, generation after generation. This can be observed in Figure 2 below, in which we assume a constant rate of growth of population (through time and equal for all classes) $g_e$, and in which we assume average values of the parameters (as in Table 1).

To illustrate this process let us start at the extreme left of Figure 2. For all points to the left of B the dynasties are initially very low risk-averse and hence, following the plan described in the preceding Section II, will consume relatively little and accumulate at a rate higher than population growth. But, as time passes, the dynasties, by increasing their wealth, will tend to become less risk-averse; consumption will hence be further reduced so increasing the accumulation rate and converging towards the point A.

**Figure 2**
The Dynamics of the Accumulation Process Under Uncertainty With a Single Risky Capital Market

![Figure 2 Diagram]

where: $g^*$ is the rate of growth of mean wealth;
$\hat{g}^*$ is the rate of growth of mean wealth under diminishing marginal returns;
$g_e$ is the constant rate of growth of population;
$r-\delta$ is the value of $g^*$ for $1-a = R_R(B) = 1$.

The inverse will happen to those who start off under the $g_e$ horizontal line: they accumulate at a rate which is too low and hence will end up at point C. At C the accumulation rate is just enough to guarantee a constant positive bequest to
future generations; here we have an equilibrium point beyond which the dynasty will not go.

Those who start off to the right of C, above the $g_e$-line\(^1\), will tend to accumulate at a fast rate thus becoming, throughout the generations, less and less risk-averse. In this way they will tend to converge towards point C, beyond which they will not go.

So in the very long run we would end up with a two-class society: one class will have a relatively small bequest per capita (those converging towards point C, who will be rather risk-averse) and the other class with a much larger bequest per capita (those converging towards point A, who will be very low risk-averse).

At this point the following remarks are in order, since the above mechanism is based, among other things, on the following assumptions:

1. The value of the rate of growth of population must be within the following limits:

$$g^\ast(\text{min}) < g_e < r - \delta,$$

otherwise the above process would end up with one class only (of very rich or poor people for $g^\ast(\text{min}) > g_e$ or $g_e > r - \delta$ respectively). However the limits within which $g_e$ must lie for the above conclusions to be valid (i.e. for converging towards a two-class society), for realistic values of the parameters as $r = 6\%$, $\delta = 3\%$ and $v = 0.001$, are as follows:

$$0.0072 < g_e < 0.03,$$

a reasonable assumption, satisfied in most of our economies.

2. The rate of growth of population is assumed to be constant and homogeneous for all dynasties (independently of their wealth); it may be stressed that the assumption of a differentiated fertility would mitigate or increase the inequalities in the ownership of capital. This aspect has been analysed, in a different framework, by Meade (1973).

Suppose, for instance, that the fertility of the relatively less risk-averse (i.e. on the left hand-side of point B) was to rise: in this case the dynasties which were immediately above the $g_e$-line (cf. Figure 2) would converge now towards point C. On the other hand if the dynasties with more risk-aversion would have a larger number of children they would be able to accumulate somewhat larger properties, since the point C would be shifted to the right.

3. The dynasties are intergenerationally stable and inter-class marriages do not alter the mechanism. This is a rather important assumption; but its relaxation, intuitively, shouldn't alter much the results obtained, at least in the very long run.

\(^1\) For average values of the parameters those who start off to the right of C will be a small minority with a very high risk-aversion.
In Figure 2 we also give an approximate line for the rate of growth of the mean wealth under the assumption of diminishing marginal returns, which is somewhat lower than $g^*$; under this assumption the rate of return of the wealthiest dynasties tends to decline. The conclusion drawn for the $g^*$ line (which does not imply diminishing marginal returns) remains however always valid.

IV. Conclusions

In this paper we have studied the effects of interest uncertainty (i.e. of risk-aversion and variance of the risky rate of return) on the variables of a stochastic life-cycle model without portfolio choice. The variables on which we have focused our attention are the consumption rate ($c^*$) and the rate of growth of mean wealth ($g^*$).

Our analysis has shown that while the variance has a definite effect on the two variables (negative on $c^*$ and positive on $g^*$), the risk-aversion measure does have an ambiguous influence on them. More precisely as risk-aversion increases, optimal consumption is first increased, attains a maximum (for average values of the risk-aversion) and then decreases slowly reaching zero for very high values of the risk-aversion. The inverse applies for $g^*$, the rate of growth of the mean wealth, which attains a minimum for median values of the risk-aversion. Hence the main conclusion, in this context, is that uncertainty does not have an unambiguous effect on the optimal variables of the model.

Looking at the long term dynamics of the model, in Section III we have shown that under some particular (but not very restrictive) assumptions our system would converge towards a two-class society: one class with a relatively small per capita financial capital stock, and another with an everincreasing one. This result, which rejoins the old classical proposition of a two-class society, is however the outcome of optimal behaviour in a model which does not consider labour income.

References

Zusammenfassung

Die Wirkungen von unsicheren Zinserwartungen in einem Lebenszyklus-Modell


Résumé

Les implications de l'incertitude relative au taux d'intérêt dans un modèle à cycles vitaux

Le but de cet article est d'étudier les conséquences de l'introduction de l'incertitude relative au taux d'intérêt sur les choix de consommation et d'épargne dans le cadre d'un modèle à cycles vitaux. La forme de l'incertitude considérée ici est une version simplifiée de celle en temps continu de Markowitz/Merton/Flemming. Notre analyse montre que une augmentation de la variance du taux d'intérêt provoque une diminution du taux de consommation, tandis que l'aversion au risque a un effet indéterminé sur ce même taux. Quant aux propriétés à long terme du modèle, nous montrons que l'incertitude peut être à l'origine de la formation de deux classes de détenteurs de fortune financière.

Summary

The Effects of Interest Uncertainty in a Life-Cycle Model

This paper explores the consequences of the introduction of interest uncertainty on the consumption- and saving-plans of a simple life-cycle model. The uncertainty structure that we consider here is a simplified version of the Markowitz/Merton/Flemming continuous-time approach. Our analysis shows that the variance of the rate of interest has a negative effect on the consumption rate, while the risk-aversion has an ambiguous influence on it (first positive and then negative as the risk-aversion increases). Looking at the long term properties of the model we show that uncertainty may be relevant in generating two distinct classes of financial capital owners.