Optimal Currency Hedging, Export, and Production in the Presence of Idiosyncratic Risk

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I. INTRODUCTION

Since the advent of floating exchange rates among the major industrial countries in 1973, foreign exchange rates have fluctuated extensively. For risk-averse economic agents with future commitments in foreign currencies, exchange rate risk has increased substantially. At the same time, the volume of trade in foreign currency futures, forwards, and options has grown considerably. This raises the question whether and how these instruments may be used to hedge exchange rate risk. This question has been answered for the problem of a risk-averse exporting firm facing exchange rate uncertainty by, e.g., HOLTHAUSEN (1979), KATZ and PAROUSH (1979), FEDER, JUST, and SCHMITZ (1980) and ZILCHA and ELDOR (1991). These models suggest that under certain conditions the introduction of currency forwards eliminates the adverse effects of exchange rate uncertainty.

However, only few models deal with more than one type of risk. Exceptions are BENNINGA, ELDOR, and ZILCHA (1985) and KAWAI and ZILCHA (1986) who consider price risk in addition to exchange rate risk. BROLL and ZILCHA (1991) include interest rate uncertainty. In these models, however, the additional risk can be traded through commodity or interest rate futures. If these hedging instruments are missing, changes in output provide partial insurance against this risk. This corresponds to the classical result of SANDMO (1971).

The present paper extends the existing literature by introducing an additional source of risk that is independent of the exchange rate and that cannot be traded. Independent risk is called idiosyncratic risk. The idiosyncratic risk captures all kinds of uncertainty that are not hedgeable and independent of the firm’s export activities, for example unexpected changes in fixed costs, e.g., environmental restrictions, or independent risks from domestic activities. It is shown that the existence of idiosyncratic risk does not change the qualitative relation between forward market characteristics and the optimal hedging position, but optimal speculative positions will be smaller due to idiosyncratic risk. Additionally, this paper provides an application of the concept of prudence as developed by KIMBALL (1990) and extended by FRANKE, STAPLETON, and SUBRAHMA-
NYAM (1992) on international economics and microeconomics of the firm under uncertainty.

The paper is organized as follows: Section II presents a two-date model of a competitive exporting firm which has access to currency forward markets. Section III introduces the concept of prudence and provides a reformulation of the firm’s optimization problem. Section IV characterizes the optimal hedging decision. In Section V, comparative static effects of changes in idiosyncratic risk and risk aversion are analyzed. Section VI concludes the paper.

II. THE MODEL

Consider a risk-averse firm which exports its entire output q. The price on the foreign market, p, is denominated in foreign currency. Additionally, the firm can take a position F in the currency forward market. A positive F is defined as the amount of foreign exchange sold in the forward market, f is the forward exchange rate. Since both, the output market and the forward market are assumed to be competitive, p and f are exogenously given constants. Decisions on output and the forward position are made at date zero. At date one, the firm sells its output and converts its revenue into home currency at the stochastic exchange rate 2. The distribution of 2 is known to the firm. The cost function c(q) is strictly convex, increasing, and differentiable, i.e., c'(q) > 0, c''(q) > 0. The firm faces an additional risk e which is assumed to be stochastically independent of the exchange rate 2. Furthermore, e can neither be hedged nor diversified away. e is assumed to have zero mean and a standard deviation of one. α is a non-negative constant. In the absence of idiosyncratic risk, α equals zero. An increase in α implies a mean-preserving spread in idiosyncratic risk in the sense of ROTHSCHILD and STIGLITZ (1970). The firm’s profit, w, is defined in local currency.

\[
\hat{w} = \hat{s}pq + F(f-\hat{s}) - c(q) + \alpha \hat{e}
\]

(1)

where g( 2 ) denotes the profit without the idiosyncratic risk component. The firm maximizes the expected value of its von Neumann-Morgenstern utility function U by choosing its output and the hedging position in the forward market. Thus, the optimization problem is

1. Since basis risk does not exist in a two-date world, there is no distinction between forward and futures contracts.
2. A tilde denotes a random variable.
4. Costs c(q) are compounded to date one.
\[ \max_{q,F} \mathbb{E}[U(\tilde{w})] \] (2)
\[ \text{s.t. equation (1),} \]
where the expectation concerns \( \tilde{s} \) and \( \tilde{e} \). The necessary and sufficient first order conditions for an interior solution are
\[ \mathbb{E}[U'(\tilde{w}) (\tilde{s} p - c'(q))] = 0, \] (3)
\[ \mathbb{E}[U'(\tilde{w}) (f - \tilde{s})] = 0, \] (4)
respectively. It is assumed that these conditions always imply a positive output.

III. PRUDENCE, PRECAUTIONARY PREMIUM, AND DERIVED UTILITY

In order to derive results for the present case of two random elements in profit, the concepts of prudence and the so-called precautionary premium are useful tools. Thus, they are presented briefly in this section.

Following Kimball (1990), the index of absolute prudence is defined as
\[ b(w) = \frac{-U'''(w)}{U''(w)}. \] (5)

"The term ‘prudence’ is meant to suggest the propensity to prepare and forearm oneself in the face of [here: additional] uncertainty" (Kimball (1990)). The precautionary premium \( \Psi(s) \), conditional on \( s \), for the idiosyncratic risk \( \tilde{e} \) is defined by the relation
\[ \mathbb{E}[U'(w(\tilde{s}, \tilde{e})) | s] = U'(\mathbb{E}[w(\tilde{s}, \tilde{e}) | s] - \Psi(s)) \]
\[ = U'(g(s) - \Psi(s)). \] (6)

The analogy to the well-known coefficient of absolute risk aversion defined by Pratt (1964) and Arrow (1965),
\[ a(w) = \frac{-U''(w)}{U'(w)}, \] (7)
and the risk premium \( \pi(s) \), conditional on \( s \), defined by
\[
E[U(w(\tilde{s}, \tilde{\epsilon})) | s] = U(E[w(\tilde{s}, \tilde{\epsilon}) | s] - \pi(s)) \\
= U(g(s) - \pi(s))
\]

is obvious. As the ARROW-PRATT risk premium depends on \( \alpha \) and \( s \) through \( g(s) \), so does the precautionary premium. Thus, \( \Psi = \Psi(g(s), \alpha) \) for a given \( s \). The ARROW-PRATT risk premium measures the effect of \( \tilde{\epsilon} \) on utility itself while the precautionary premium is concerned with the effect on marginal utility. The coefficient of absolute prudence can be interpreted as the coefficient of risk aversion defined on negative marginal utility. Since it is marginal utility that determines the optimal value(s) of the firm's decision variable(s), the precautionary premium and the coefficient of prudence play an important role for the "sensitivity of the optimal choice of \([\text{the decision variable}(s)]) \text{ to risk}" (KIMBALL (1990)). For further interpretation, see KIMBALL (1990) and, for a combination with idiosyncratic risk, FRANKE, STAPLETON, and SUBRAHMANYAM (1992).

Furthermore, the following assumptions are imposed on the utility function: Absolute risk aversion and absolute prudence are positive and decreasing\(^5\). A subclass of the hyperbolic absolute risk aversion (HARA) class of utility functions satisfies these assumptions\(^6\). The assumptions imply alternating signs of the first four derivatives of the utility function\(^7\), i.e., \( U' > 0, U'' < 0, U''' > 0, U'''' < 0 \). This is closely related to the concept of "proper risk aversion" as proposed by PRATT and ZECKHAUSER (1987). Roughly speaking, "proper risk aversion" means that two or more independent risks each of which is undesirable by itself cannot be desirable together. The class of utility functions that exhibit "proper risk aversion" includes those infinitely differentiable functions whose derivatives alternate in sign\(^8\).

Following FRANKE, STAPLETON, and SUBRAHMANYAM (1992), the precautionary premium \( \Psi \) can be used to derive a utility function \( V \) from \( U \). Using the definitions of \( \pi(g(s), \alpha) \) and \( \Psi(g(s), \alpha) \), \( V \) is defined by

\[
E[U(g(\tilde{s}) + \alpha \tilde{\epsilon}) | s] = E[V(g(\tilde{s}) - \pi(g(\tilde{s}), \alpha)) | s], \\
E[U'(g(\tilde{s}) + \alpha \tilde{\epsilon}) | s] = E[V'(g(\tilde{s}) - \pi(g(\tilde{s}), \alpha)) | s] \cdot \left[ 1 - \frac{d \pi}{dg} \right]
\]

5. The assumptions on absolute risk aversion trace back to ARROW (1965). The assumptions concerning absolute prudence are due to KIMBALL (1990) and FRANKE, STAPLETON, and SUBRAHMANYAM (1992).

6. The HARA class is given by \( U(w) = (1-\gamma)/\gamma [A + w/(1-\gamma)]^\gamma \), where \( A \) is constant. Additionally, \( A + w/(1-\gamma) \) is required to be positive, implying that \( \gamma \) must be bounded from below. The subclass satisfying the above assumptions imposes the restriction \( 1 > \gamma > -\infty \) on \( \gamma \). FRANKE, STAPLETON, and SUBRAHMANYAM (1992) provide a detailed analysis of the precautionary premium in this HARA subclass.

7. This can easily be seen by differentiating the definitions in (5) and (7).

8. However, alternating signs of the derivatives do not necessarily imply decreasing absolute risk aversion and prudence.
The utility function $V(g - \pi)$ and the marginal utility function $V'(g - \Psi)$ can be interpreted as a derived utility function (NACHMAN (1982)) and a derived marginal utility function, respectively. "Derived" means the incorporation of $\bar{e}$ into the utility function indirectly via $\pi$ and into the marginal utility function via $\Psi$.

Decreasing absolute risk aversion of $U$ is equivalent to $d\pi/dg < 0$ (PRATT (1964)). The assumption of positive marginal utility, $U' > 0$, and the definition in equation (10) imply $V' > 0$. FRANKE, STAPLETON, and SUBRAHMANYAM (1992) have shown that under the above assumptions on $U$, the derived utility function $V$ exhibits risk aversion. Thus, $V'' < 0$.

Idiosyncratic risk changes the utility function from $U$ to $V$ such that $V$ retains the usual properties $V' > 0$ and $V'' < 0$. The following analysis is based on the derived utility function $V$.

IV. OPTIMAL HEDGING AND PRODUCTION

In this section, optimal hedging and production decisions are presented. Using the derived utility function $V$, the first order conditions (3) and (4) can be rewritten as follows. For notational simplicity, $V(g(\bar{s}) - \Psi(g(\bar{s}), \alpha))$ is denoted by $V'(\bar{s})$.

$$E[V'(\bar{s}) (\bar{s}p - c'(q))] = 0, \quad (11)$$
$$E[V'(\bar{s}) (f - \bar{s})] = 0. \quad (12)$$

Combining these conditions leads to a well-known separation result.

**Proposition 1: (Separation)** The firm chooses its optimal output $q$ following

$$pf = c'(q). \quad (13)$$

Thus, the firm's decision about its optimal export production is affected neither by its risk aversion nor by the distribution of the foreign exchange rate $\bar{s}$ and the idiosyncratic risk $\bar{e}$.

**Proof:** Since $E$ is a linear operator, condition (11) can be rewritten as

$$p E[\bar{s}V'(\bar{s})] - c'(q) E[V'(\bar{s})] = 0. \quad (14)$$

Similarly, from (12)

$$f E[V'(\bar{s})] - E[\bar{s}V'(\bar{s})] = 0. \quad (15)$$
Substituting (15) into (14) yields

\[ p f E[V'(\bar{s})] - c'(q) E[V'(\bar{s})] = 0. \]  

(16)

Hence equation (13) follows. Q.E.D.

This separation theorem has been proposed in the absence of idiosyncratic risk by ETHIER (1973), DANTHINE (1978), KATZ and PAROUSH (1979) and others. Proposition 1 is a variation of the classical FISHER-Separation (see, e.g., COPELAND and WESTON (1988)).

Separation can be interpreted as a kind of no-arbitrage condition on the individual firm's level: As long as marginal cost, \( c'(q) \), is lower than the certain revenue in domestic currency, \( pf \), the firm can earn a riskless profit. Non-satiation guarantees that the firm would use this "arbitrage opportunity" until it disappears at the level of production which is implied by equation (13).

Since the proposition completely separates expectations and preferences from the output decision, it is not surprising that the separation theorem still holds in the presence of idiosyncratic risk. Thus, neither exchange rate uncertainty nor the existence of idiosyncratic risk have the adverse effect on output which SANDMO (1971) had addressed.

Proposition 1 does not hold if the "arbitrage opportunity" is not riskless. This occurs if output is stochastic as modeled in, e.g., ROLFO (1980). Another reason for the separation theorem not to hold is basis risk caused by an imperfect time match between the marketing date of the output and the delivery date of the futures contract; this is considered by BATLIN (1983) and PAROUSH and WOLF (1989).

According to the following proposition, the relation between the forward rate and the expectation about the future spot rate plays an important role for the optimal forward position. The difference \( E\bar{s} - f \) can be regarded as the risk premium in the forward market.

**Proposition 2:** (Optimal hedging) The firm hedges its output completely if and only if the forward market is unbiased, \( f = E\bar{s} \). The firm underhedges if and only if the forward market exhibits a positive risk premium (backwardation), \( E\bar{s} > f \). Finally, the firm overhedges if and only if the forward market can be characterized by a negative risk premium (contango), \( E\bar{s} < f \).

**Proof:** HOLTHAUSEN (1979).

9. The proof can be based on \( U(g) \) in this case.

10. For conflicting empirical results concerning the influence of exchange rate uncertainty on international trade see, e.g., CUSHMAN (1988).
This result has been stated by Holthausen (1979), Feder, Just, and Schmitz (1980) and others in the situation of $\alpha = 0^{11}$. An unbiased forward market offers the firm some kind of costless insurance. Due to risk aversion, the firm insures its currency risk completely if the risk premium equals zero. Furthermore, if the risk premium is positive (backwardation), the firm insures only partially. This can also be interpreted as speculation if full hedging is used as a reference. If the firm underhedges, it speculates on its expectation that the future spot price will be higher than the forward price. The reverse interpretation applies to the case of overhedging in which the firm receives a positive risk premium from the market.

It turns out from Proposition 2 that the risk premium in the forward market is the crucial determinant of the optimal hedging position. Additionally, it can be easily verified that the qualitative relation between the risk premium and the optimal hedging position is the same as in the absence of idiosyncratic risk$^{12}$. But the amount of foreign exchange which is sold forward is not independent of idiosyncratic risk. This is shown in the next section.

V. INCREASING IDIOSYNCRATIC RISK AND HEDGING

Since higher idiosyncratic risk makes a risk-averse decision maker more sensitive to additional risk, e.g., currency risk, he will be willing to pay more for a reduction of this additional risk. In other words, for a given risk premium in the forward market, the agent will insure a higher amount of his hedgeable risk. Thus, it is plausible to presume that the size of the exporter’s speculative position will decrease due to the introduction of or an increase in idiosyncratic risk. This is claimed in the following central proposition of the paper.

**Proposition 3:** (Increasing idiosyncratic risk) An increase in idiosyncratic risk induces the firm to reduce its speculative position. That is, an overhedging firm will reduce its foreign currency forward sales, an underhedging firm will increase its forward sales. The firm which fully hedges will not alter its forward position.

Proof: The proof consists of two steps. First, it is shown that higher idiosyncratic risk is equivalent to higher risk aversion of the derived utility function $V$. Consider two firms, identical with respect to all parameters except of $\alpha$. $U$ is the same for both firms. Let firm 1 face more idiosyncratic risk than firm 2, $\alpha_1 > \alpha_2$. Franke, Stapleton, and Subrahmanyam (1992) have shown that the derived utility function exhibits higher risk aversion if idiosyncratic risk is higher. Thus, the derived utility function of firm 1, $V_1$, indicates that this firm is more risk averse than firm 2 whose preferences are

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$^{11}$ For an unbiased forward market or $\alpha = 0$ the proof can based on $U(g)$.
$^{12}$ This can be seen by deriving the result of Proposition 2 for $\alpha = 0$. 
summarized by $V_2$. Therefore $V_1$ can be viewed as a concave transformation of $V_2^{13}$, i.e.,

$$V_1 = k(V_2) \text{ with } k' > 0 \text{ and } k'' < 0. \quad (17)$$

The second part of the proof can be adapted from PRATT (1964) who shows that higher risk aversion implies a smaller speculative position, i.e., $pq > F_1 > F_2$ for the case of backwardation and $pq < F_1 < F_2$ in the contango case. Q.E.D.

Proposition 3 illustrates KIMBALL's (1990) statement that the precautionary premium can be used to analyze the sensitivity of a decision variable in response to risk, because idiosyncratic risk and the precautionary premium are directly related by $\Psi = b \alpha^2/2$ for a small idiosyncratic risk (FRANKE, STAPLETON, and SUBRAHMANYAM (1992)). However, Proposition 3 does not state that the exporting firm will change the sign of its speculative position, i.e., an underhedging firm will not become an overhedging firm or vice versa.

It follows directly from Proposition 3 that the introduction of idiosyncratic risk causes the firm to reduce its speculative position. Due to the monotonicity of this position in $\alpha$, it is reasonable that the firm's speculative position approaches zero as idiosyncratic risk becomes very high. That is, in the limit the firm enters into a full hedging position due to high idiosyncratic risk, irrespective of its exchange rate expectations.

Additionally, the second part of the proof of Proposition 3 implies that an increase in risk aversion causes the firm to reduce its speculative position for a given $\alpha$.

VI. CONCLUDING REMARKS

This paper provides a further application of the concepts of prudence and the precautionary premium, used by KIMBALL (1990) for analyzing the insurance effects of precautionary saving. In the present context, the reduction in the speculative position may also be interpreted as partial insurance against idiosyncratic risk since the sensitivity of the firm's profit due to changes in the exchange rate is reduced. It is worth noting that, in contrast to most of the literature, changes in the distribution of non-hedgeable risk have an impact on the optimal level of insurance in the forward market even if the non-hedgeable risk is stochastically independent of the insurable risk\textsuperscript{14}. Since non-hedgeable risks appear to play an important role in the real world, further applications should be undertaken.

13. See PRATT (1964). Note that $V_1$ and $V_2$ differ only due to $\alpha$, but not due to the (ordinary) utility function $U$.

14. Implications of this result for the pricing of state contingent claims are analyzed by FRANKE, STAPLETON, and SUBRAHMANYAM (1992) in a complete markets context.
The present example incorporates idiosyncratic risk into the optimization problem of a risk-averse exporting firm. It is shown that the separating property of forward markets and the qualitative characteristics of the optimal forward position are robust to the introduction of idiosyncratic risk. Thus, the production decision is affected neither by the firm's risk aversion or prudence nor by the distribution of the exchange rate or the idiosyncratic risk. This is somewhat in contrast to other examples for "missing markets" as addressed in BENNINGA, ELDOR, and ZILCHA (1985) and KAWAI and ZILCHA (1986).

The qualitative characteristics of the optimal hedging position remain the same if idiosyncratic risk is introduced: If the forward market shows backwardation, the risk averse exporter still underhedges its foreign revenue. In the case of contango, the exporter still overhedges. But the speculative component in the firm's optimal forward position is shown to be smaller under idiosyncratic risk provided that the exporter is decreasingly risk-averse and decreasingly prudent. Although idiosyncratic risk is assumed to be independent from the currency risk, it influences the optimal forward position.

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SUMMARY

This paper analyzes the behavior of a risk-averse exporting firm facing exchange rate uncertainty in the presence of forward markets. The existing literature on optimal hedging and production rules is extended by allowing for idiosyncratic risk. The paper provides an application of recent concepts in expected utility theory concerning optimal decisions in the presence of more than one risk (prudence, precautionary premium). Important results (separation theorem, full hedging theorem) still hold, but optimal speculative positions are smaller due to the existence of idiosyncratic risk.

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RÉSUMÉ

Cet article traite le comportement des entreprises exportantes qui sont risque-averses et peuvent couvrir leurs risques de taux de change sur les marchés à terme. L’article étend la littérature de façon que ces entreprises envisagent – à part le risque de taux de change – un risque de perte “idiosyncratique”. Des concepts récents de la théorie espérance-utilité (prudence, prime de précaution) sont appliqués pour pouvoir analyser plusieurs risques qui apparaissent simultanément. Les résultats importants de la séparation de la décision de production de celle de la couvrance restent valable. D’autres résultats, par contre, indiquent que le risque de perte “idiosyncratique” mène à une réduction des positions de devises spéculatives.