Dividend Growth Uncertainty and Stock Prices

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1. INTRODUCTION

One of the striking phenomena in recent stock market history is the continuous rise in stock prices. Over the last few years, price-earnings ratios have risen to a level seldom seen in history. The "new economy" literature has provided a number of explanations for this phenomenon. One of them is a high expected long-term growth. Indeed, proponents of the new economy argue that the economy, especially in the U.S., has entered a new era of intensified competition and rising productivity growth, thus leading to higher stock prices.¹ Another strand of the "new economy" literature has focused on how well reported firm earnings reflect true profitability. Nakamura (1999) argues that, in the new economy, a large proportion of corporate investment occurs in the form of expenditures leading to the development of intangible assets. Since accounting conventions do not recognize these expenditures as investments, corporate profits are understated. Nakamura argues that if these factors are taken into account, the recent U.S. economic and financial performance is easier to explain.

Common to all these models is the idea that underlying firm profitability is high. High stock valuations and price-earnings ratios are achieved either because firm profits are already high (although understated by accounting data), or the rate of expected growth in profits is high. The analysis in this paper demonstrates that high stock valuations may arise not only as a result of high current profits or high future growth, but also as a result of high uncertainty about future dividend growth per se.

In order to establish this result, we model a simple representative-agent economy with a single risky asset available for investment. The representative agent's only income is the dividend stream paid by the asset. The agent does not know the expected growth rate in dividends, and must therefore estimate it from the historical path of dividends. Our analysis shows that the higher the agent's (parameter) uncertainty about the expected growth in dividends, the higher the asset price and the price-earnings ratio. Thus,

¹ See Browne (1999) for an overview of this literature and a discussion of the main issues.

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dividend growth uncertainty can explain the current valuations achieved on stock markets. Moreover, our analysis demonstrates that parameter uncertainty can lead prices to over- or underreact to changes in dividends, depending on whether the representative agent is more or less risk-averse than the log-utility investor.

The analysis in this study is related to a number of recent papers in the financial economics literature. Brennan (1998a) considers the effect of uncertainty about an asset’s expected return on investors’ optimal dynamic asset allocation and finds that this effect can be substantial for time horizons as short as ten years. Brennan and Xia (1999) show that by distinguishing between the cash flows received by equity holders and aggregate consumption, and by allowing for uncertainty and learning about a stochastic but unobservable dividend growth rate, it is possible to generate a representative agent model which yields a stock price series whose first two moments match those of the historical price series. In a related paper, Xia (2000) analyzes the effect of parameter uncertainty on dynamic asset allocation. She finds that parameter uncertainty induces a state-dependent hedge portfolio demand that may increase or decrease with the investor’s time horizon. Barberis (2000) considers the effect of predictability in asset returns on optimal portfolio choice for investors with long horizons. He shows that even after incorporating parameter uncertainty, predictability in returns has a significant impact on investors’ portfolio allocations. These choose to allocate substantially more to stocks, the longer their time horizon.

As this discussion has made clear, all of these papers are primarily concerned with the effect of parameter uncertainty on optimal asset allocation strategies and equilibrium risk premia. In contrast, it is the effect of parameter uncertainty on the level of stock prices which constitutes the focus of this paper.

Although it is cast in terms of dividend uncertainty for a single risky asset, the model presented below can be given a broader, macroeconomic interpretation. In this interpretation, the dividend stream represents the total consumption stream available in the economy, and the investor’s uncertainty about the future growth in dividends measures “macroeconomic uncertainty,” i.e. the extent to which future economic growth is known. Similarly, the price of the risky asset can be viewed as the total market capitalization in the economy, and the single risky asset’s price-earnings ratio (P/E) as the market-wide P/E.

Our reason for focusing on dividend growth uncertainty in this paper is two fold. First, there is considerable evidence that uncertainty is high at present. The current debate of whether a “new economy” exists at all, i.e. whether recent growth – especially in the U.S. – is a temporary or a permanent phenomenon (see Browne 1999 for an overview of this discussion) and the controversy about the consequences of advances in information technology for productivity and long-term growth are symptoms of this situation. Hand in hand with this uncertainty about the macroeconomy’s prospects comes a high uncertainty about the overall level of current profits and the future evolution of dividends in the economy. As far as the author of this paper can ascertain, however, the consequences of this uncertainty for asset prices remain largely unanalyzed and current stock
market valuations still constitute a puzzle to most economists. Second, in showing that uncertainty about future growth, rather than a bright future per se, can lead to high stock market valuations, we can avoid drawing wrong inferences about our future from current market valuations.

The paper is organized as follows. Section 2 presents the model. Section 3 computes the equilibrium price using state-price deflator techniques. Section 4 discusses the properties of the equilibrium price process and presents the main results of the paper. Section 5 concludes.

2. THE MODEL

Consider a continuous-time representative-agent economy with a single firm.2 The firm produces a single good. It is completely financed by equity and has one share outstanding. The firm pays a dividend to its shareholder at a rate $x_t$ at time $t$. Suppose that the dividend process follows

$$dx_t = \mu x_t dt + \sigma x_t dB_t,$$

where $\mu$ is the constant instantaneous increase in expected dividends, $\sigma$ denotes the dividend process’ constant instantaneous volatility, and $B$ denotes a standard Brownian motion. Suppose, however, that the agent doesn’t know the true mean $\mu$, but rather must estimate it from past data. We assume that at initial time $t = 0$, the agent has the following prior information on $\mu$: he knows that $\mu$ is normally distributed with mean $m_0$ and variance $V_0 = E((m_0 - \mu)^2)$. The investor has no additional prior information on $\mu$. He only has the filtration $F_t = \sigma(x_s, s \leq t)$. However, the parameter $\sigma$ is assumed to be known.

From filtering theory, as new dividend information arrives, the agent updates his estimate $m_t$ of the mean growth in dividends $\mu$ using the relationship3

$$dm_t = \frac{V_t}{\sigma} dB_t,$$

where

2. As shown in Rubinstein (1974), a representative agent can be defined if all agents have identical beliefs and similar (e.g., constant relative risk aversion) utility functions. In using a representative-agent setting, however, the analysis below abstracts from life-cycle considerations (which would best be analyzed in an overlapping generations model) and from the effect of heterogeneous beliefs on equilibrium asset prices.

\[ d\bar{B}_t = dB_t + \frac{\mu - m_t}{\sigma} dt. \]  

(3)

Using the fact that
\[ d\bar{B}_t = dB_t + \frac{\mu - m_t}{\sigma} dt = \frac{1}{\sigma} \left( \frac{dx_t}{x_t} - \mu dt \right) + \frac{\mu - m_t}{\sigma} dt = \frac{1}{\sigma} \left( \frac{dx_t}{x_t} - m_t dt \right), \]  

(4)

one can rewrite (2) as
\[ dm_t = \frac{V_t}{\sigma} dB_t = \frac{V_t}{\sigma^2} \left( \frac{dx_t}{x_t} - m_t dt \right). \]  

(5)

Thus, the investor updates his estimate \( m_t \) by multiplying the "surprise" component of the change in dividend, \( \left( \frac{dx_t}{x_t} - m_t dt \right) \), with the term \( \frac{V_t}{\sigma^2} \), which is a measure of his relative uncertainty about \( \mu \). From the agent's viewpoint, this partially observed economy with constant mean expected growth in dividends \( \mu \) is equivalent to a perfectly observed economy with stochastic, time-varying mean expected growth in dividends \( m_t \). Note that the instantaneous change in dividends \( \frac{dx_t}{x_t} \) and the investor's estimate of \( \mu \) are perfectly correlated. Thus, the investor's expectations formation is "extrapolative" (see Brennan 1998a): when \( \frac{dx_t}{x_t} > m_t dt \), the investor revises his estimate of \( \mu \) upwards. When \( \frac{dx_t}{x_t} < m_t dt \), he revises his estimate of \( \mu \) downwards.

In expressions (2) and (5), \( V_t = E_t((m_t - \mu)^2) \) denotes the mean square error of \( m_t \) and has dynamics
\[ dV_t = -\frac{V_t^2}{2} dt. \]  

(6)

This latter equation, together with the initial condition \( V_0 \), implies
\[ V_t = \frac{1}{\frac{1}{V_0} + \frac{1}{\sigma^2} t}. \]  

(7)

The mean square error \( V_t \) is a measure of the agent's uncertainty about the true value of \( \mu \). It can also be thought of as a measure of the (inverse) quality of the agent's information. When \( V_t = 0 \), \( \mu \) is perfectly known and the economy is a complete-information economy. When \( V_t \) is high, the agent's uncertainty about future dividend growth is high and he undertakes large revisions of \( m_t \) as new dividend information comes in.

To gain some intuition for the agent's updating behavior as presented in equations (2) and (6), consider the case in which the agent only has diffuse prior information on \( \mu \), i.e. \( V_0 = \infty \). Then, given the fact that
\[ x_t = x_0 \cdot \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \int_0^t dB_s \right), \]  

or

\[ \ln(x_t) = \ln(x_0) + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma \int_0^t dB_s, \]  

his best estimate for \( \mu \) at time \( t \) will be

\[ m_t = \frac{\ln(x_t) - \ln(x_0)}{t} + \frac{\sigma^2}{2}. \]  

The mean square error of this estimate, \( V_t \), will be

\[ V_t = E((m_t - \mu)^2) = E \left( \left( \frac{\ln(x_t) - \ln(x_0)}{t} + \frac{\sigma^2}{2} - \mu \right)^2 \right) \]

\[ = E \left( \frac{\sigma^2}{t^2} \int_0^t dB_s \right)^2 \]

\[ = \frac{\sigma^2}{t^2} E \left( \int_0^t 1 ds \right) = \frac{\sigma^2}{t}, \]  

where the first equality on the last line results from the Ito Isometry.\(^4\) To determine how our agent updates his estimate \( m_t \) of the instantaneous increase in expected dividends \( \mu \), one can apply Ito’s formula to (10), yielding the following expression for the dynamics of \( m_t \):

\[ dm_t = \frac{d \ln(x_t)}{t} - \frac{\ln(x_t) - \ln(x_0)}{t^2} dt \]

\[ = \left( \frac{\mu - \sigma^2/2}{t} dt + \sigma dB_t \right) - \frac{\ln(x_t) - \ln(x_0)}{t^2} dt \]

\[ = \left( \frac{\mu - m_t}{t} dt + \sigma dB_t \right) = \frac{\sigma}{t} d \tilde{B}_t = \frac{V_t}{\sigma} d \tilde{B}_t, \]  

where \( d \tilde{B}_t = dB_t + \frac{\mu - m_t}{\sigma} dt \). Equation (12) is thus consistent with the general filtering expression (2). To determine how the mean square error of his estimate decreases through time as our agent updates his beliefs, one can differentiate (11) with respect to \( t \), yielding

\[ dV_t = -\frac{\sigma^2}{t^2} dt, \]  

which can also be written as $dV_t = -\frac{V^2}{\rho} dt$, a result again consistent with the general filtering equation (6).

Suppose that our representative agent has a time-separable, additive utility function defined by

$$U(c) = E\left( \int_t^T e^{-\rho s} u(c_s) \, ds \right),$$  \hspace{1cm} (14)

where $\rho$ is nonnegative and $u$ is concave in current consumption $c_s$. In order to obtain closed-form valuation equations, in the following, we assume that the representative investor has power utility,

$$U(c) = E\left( \int_t^T \frac{c^a}{\alpha} e^{-\rho s} \, ds \right).$$  \hspace{1cm} (15)

3. THE EQUILIBRIUM PRICE

This section computes the equilibrium share price at time $t$, $S_t$. As noted in the introduction, since there is only one risky asset, the share price can be understood as a proxy for overall market capitalization. Our aim is to determine how this market capitalization changes with the degree of parameter uncertainty, $V_t$.

The price of the share can be derived using the following equilibrium argument. In equilibrium, the representative agent must hold one share. Moreover, since he has no other source of income, he is restricted to consume his current dividend income, $c_t = x_t$. Hence, the state-price deflator $\pi_t$ is given by

$$\pi_t = e^{-\rho t} u'(x_t) = e^{-\rho t} x_t^{\alpha-1},$$  \hspace{1cm} (16)

and the equilibrium price of the share is given by

$$S_t = \frac{1}{\pi_t} E\left( \int_t^T \pi_s x_s \, ds \bigg| F_t^x \right) = x_t^{1-\alpha} E\left( \int_t^T e^{-\rho(s-t)} x_s \, ds \bigg| F_t^x \right).$$  \hspace{1cm} (17)

It is worth noting that the state-price deflator (16) and the equilibrium pricing relationship (17) are unaffected by incomplete information.\(^5\) However, both the estimated expected growth in dividends $m_t$ and parameter uncertainty $V_t$ affect the equilibrium share price $S_t$ through their effect on the conditional distribution of future dividends, $x_s$.

Applying Fubini's theorem, (17) can be rewritten as

\(^5\) See Honda (1997) for an example of equilibrium asset pricing under incomplete information using state-price deflator techniques.
In order to compute this expectation explicitly, one can make use of the fact that $x^a = \exp(\alpha \ln(x))$. Now, as shown in the Appendix, $\ln(x_s)$ is normally distributed with mean

$$E_t(\ln(x_s)) = \ln(x_t) + \left(m_t - \frac{1}{2} \sigma^2\right)(s - t)$$

(19)

and variance

$$\text{Var}_t(\ln(x_s)) = \sigma^2(s - t) + V_t(s - t)^2.$$  

(20)

Remembering that for a normally distributed random variable $y = \ln(x)$, $E(x^a) = E(\exp(\alpha y)) = \exp(\alpha E(y) + \alpha^2 \text{Var}(y)/2)$, we have

$$E(x^a | F^x_t) = \exp \left( \alpha \left( \ln(x_t) + \left(m_t - \frac{1}{2} \sigma^2\right)(s - t) \right) \right)$$

$$+ \frac{\alpha^2}{2} \left( V_t(s - t)^2 + \sigma^2(s - t) \right).$$

(21)

Inserting this result in equation (18) yields the following expression for the equilibrium share price $S_t$:

$$S_t = x_t \int_t^T \exp \left( \alpha \left( m_t - \frac{\rho}{\alpha} - \frac{1}{2} (1 - \alpha) \sigma^2\right)(s - t) + \frac{\alpha^2}{2} V_t(s - t)^2 \right) ds,$$

(22)

and the price-earnings-ratio $\Pi_t$ is given by

$$\Pi_t = \frac{S_t}{x_t} = \int_t^T \exp \left( \alpha \left( m_t - \frac{\rho}{\alpha} - \frac{1}{2} (1 - \alpha) \sigma^2\right)(s - t) + \frac{\alpha^2}{2} V_t(s - t)^2 \right) ds.$$  

(23)

4. PROPERTIES OF THE EQUILIBRIUM PRICE

This section analyzes the properties of the equilibrium price $S_t$ in more detail. Section 4.1 analyzes the dependence of the price on expected dividend growth. Section 4.2 shows that the equilibrium price increases as uncertainty about the expected growth in dividends rises. Section 4.3 discusses the consequences of parameter uncertainty for the stock price dynamics and shows that parameter uncertainty can lead prices to over-
underreact to changes in dividends. Section 4.4 considers the agent’s hedging demand against parameter estimation risk and the equilibrium price of dividend growth rate estimation risk.

4.1 Dependence on Expected Future Dividend Growth

As can be seen by taking the partial derivative of (22),

$$\frac{\partial S_t}{\partial m_t} = \alpha x_t \int_t^T (s-t) \exp\left(\alpha \left( m_t - \frac{\rho}{\alpha} - \frac{1}{2} (1 - \alpha) \sigma^2 \right) (s-t) + \frac{\alpha^2}{2} V_t (s-t)^2 \right) ds, \quad (24)$$

as the expected growth rate of dividends $m_t$ is increased, the value of the share may both increase or decrease, depending on the value of the parameter $\alpha$. When the investor has logarithmic utility ($\alpha = 0$), the share price is independent of the expected growth in dividends $m_t$. When the investor is less risk-averse than the log-utility investor ($\alpha > 0$), a higher expected growth in dividends $m_t$ leads to a higher share price. On the other hand, when the investor is more risk-averse than the log-utility investor ($\alpha < 0$), an increase in $m_t$ leads to a decrease in the share price. This effect is depicted in Figure 1.

Figure 1: Share price $S_t$ as a function of the estimated expected growth rate in dividends $m_t$

When the investor is less risk-averse than the log-utility investor ($\alpha > 0$, solid line), the share price is increasing in $m_t$. When the investor is more risk-averse than the log-utility investor ($\alpha < 0$, dotted line), the reverse holds. (Value of the parameters: $x_t = 1$, $\alpha = \pm 0.5$, $\rho = 0.05$, $\sigma = 0.2$, $t = 0$, $T = 10$, $V_t = 0.1$)
The result that the share price can either rise or fall with the expected growth rate of dividends $m_t$ is somewhat counterintuitive. Intuition would suggest that the share price should rise when prospects for future dividends improve. This need not be the case, however. The reason is that an increase in $m_t$ has two effects:

- First, it leads to an increase in future dividends $x_s$, which is good for the share price.
- Second, when future dividends are higher, the state-price deflator $\pi_s$ is lower due to risk-aversion, which hurts prices.

Which of these two effects dominates in practice will depend on the agent's level of risk aversion. When the representative agent is less risk-averse than the log-utility investor ($\alpha > 0$), the first effect dominates, and prices rise as $m_t$ increases. When the agent is more risk-averse than the log-utility investor ($\alpha < 0$), the second effect dominates, and prices fall as $m_t$ increases.

### 4.2 Dependence on Parameter Uncertainty

Consider now the dependence of asset value on the degree of uncertainty about expected growth in dividends, $V_t$. As illustrated in Figure 2, as long as $\alpha \neq 0$, i.e. as long as the investor does not have logarithmic utility, an increase in parameter uncertainty $V_t$ has a positive effect on the share price. Formally, an application of Leibniz' rule shows that as parameter uncertainty $V_t$ increases, the value of the share rises more and more above its complete-information value:

$$\frac{\partial S}{\partial V_t} = x_t \frac{\alpha^2}{2} \int_t^T (s-t)^2 \exp \left( \alpha \left( m_t - \frac{\rho}{\alpha} - \frac{1}{2} (1 - \alpha) \sigma^2 \right) (s-t) + \frac{\alpha^2}{2} V_t (s-t)^2 \right) ds > 0. \quad (25)$$

This result is somewhat counterintuitive, since one would expect a risk-averse agent to react negatively to an increase in uncertainty about the true mean $\mu$, thus lowering the share price. However, it can be understood in several ways. The first is to remember that the underlying economy is an economy with unknown, but constant mean growth parameter $\mu$. Suppose first that $\mu$ were known. Then, the price of the share would be given by

$$S_{t,\mu} = x_t \int_t^T \exp \left( \alpha \left( \mu - \frac{\rho}{\alpha} - \frac{1}{2} (1 - \alpha) \sigma^2 \right) (s-t) \right) ds. \quad (26)$$
As parameter uncertainty $V_t$ is increased, the equilibrium price rises, both when the investor is less risk-averse than the log-utility investor ($\alpha > 0$, solid line) and when the investor is more risk-averse than the log-utility investor ($\alpha < 0$, dotted line). (Value of the parameters: $x_t = 1$, $\alpha = \pm 0.5$, $m_t = 0.1$, $\rho = 0.05$, $\sigma = 0.2$, $t = 0$, $T = 10$)

The share price is a convex function of $\mu$ both when the investor is less risk-averse than the log-utility investor ($\alpha > 0$, solid line) and more risk-averse than the log-utility investor ($\alpha < 0$, dotted line). Jensen’s inequality then implies that the share price is increasing in parameter uncertainty $V_t$. (Value of the parameters: $x_t = 1$, $\alpha = \pm 0.5$, $\rho = 0.05$, $\sigma = 0.2$, $t = 0$, $T = 10$)
This complete-information price \( S_{t,\mu} \) is depicted in Figure 3 for a range of values of \( \mu \). Note that \( S_{t,\mu} \) is convex in \( \mu \). An application of Leibniz’ rule shows that this will be the case for all parameter values:

\[
\frac{\partial^2 S_{t,\mu}}{\partial \mu^2} = x_t \int_t^T \alpha^2 (s-t)^2 \exp \left( \alpha \left( \mu - \frac{\rho}{\alpha} - \frac{1}{2} (1 - \alpha) \sigma^2 \right) (s-t) \right) ds > 0.
\] (27)

But then, since \( \mu \) is in fact a random variable, Jensen’s inequality implies that the price of the share under incomplete information must be greater than the one under complete information. Remembering that \( \mu \) is normally distributed with mean \( m_t \) and variance \( V_t \), and applying the law of iterated expectations, one obtains\(^6\)

\[
S_t = E_t(S_{t,\mu}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V_t}} \exp \left( - \frac{(\mu - m_t)^2}{2V_t} \right) \left( x_t \int_t^T \exp \left( \alpha \left( \mu - \frac{\rho}{\alpha} - \frac{1}{2} (1 - \alpha) \sigma^2 \right) (s-t) \right) ds \right) d\mu.
\] (28)

Using Fubini’s theorem and completing the squares yields

\[
S_t = x_t \int_t^T \left( \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi V_t}} \exp \left( - \frac{(\mu - m_t)^2}{2V_t} \right) d\mu \right) ds \left( x_t \int_t^T \exp \left( \alpha \left( m_t - \frac{\rho}{\alpha} - \frac{1}{2} (1 - \alpha) \sigma^2 \right) (s-t) + \frac{\alpha^2}{2} V_t (s-t)^2 \right) ds \right) d\mu.
\] (29)

which is the same value as that computed in equation (22). Thus, the fact that the share price is increasing in parameter uncertainty \( V_t \) can be considered as resulting from the fact that the complete-information share price \( S_{t,\mu} \) is convex in \( \mu \). If instead \( S_{t,\mu} \) were concave in \( \mu \), asset value under incomplete information would be lower than that under complete information.

\(^6\) The economic intuition for using iterated expectations in order to determine the equilibrium price is the following. From the perspective of the representative agent, what matters for valuation is the marginal utility of consumption in a given state of nature (the state-price deflator \( \pi_s \)), the amount of consumption in this state of nature, \( c_s = x_s \), and the probability distribution of states of nature. Given these three factors, however, it is irrelevant if a given state of nature results because of a high true mean expected growth of dividends \( \mu \) or because of a high dividend path.
An alternative way of understanding the positive dependence of the share price on parameter uncertainty is the following. Recall that the equilibrium asset price process \( S_t \) is given by

\[
S_t = \frac{1}{\pi_t} E \left( \int_t^T \pi_s x_s ds \right) ^{\alpha}.
\]

(30)

Now, an increase in parameter uncertainty \( V_t \) has two effects:

- First, as we remember from the above discussion, the conditional mean and variance of \( \ln(x_s) \) are given by
  \[
  E_t(\ln(x_s)) = \ln(x_t) + \left( m_t - \frac{1}{2} \sigma^2 \right) (s - t) \quad \text{and} \quad \text{Var}_t(\ln(x_s)) = \sigma^2 (s - t) + V_t (s - t)^2.
  \]
  Therefore, \( E_t(x_s) = x_t \exp \left( m_t (s - t) + \frac{1}{2} V_t (s - t)^2 \right) \) is increasing in parameter uncertainty \( V_t \).

- Second, an increase in parameter uncertainty \( V_t \) leads to an increase in the variance of \( \ln(x_s) \) and therefore of \( x_s \).

When \( \alpha > 0 \), the first effect tends to increase the share price and the second to decrease it. When \( \alpha < 0 \), the opposite occurs. In both cases, the positive effect dominates and the share price rises as uncertainty \( V_t \) is increased.

To gain some insight into the magnitude of the effect of parameter uncertainty on the equilibrium price-earnings ratio \( \Pi_t \), equation (23) was evaluated numerically for plausible values of the model parameters: a value for \( \alpha \) of -2, implying a relative risk aversion of 3, a time horizon of \( T = 25 \) years, a time preference parameter of \( \rho = 0.05 \), an expected dividend growth \( m_0 \) of 6% corresponding to the historical average growth in dividends on the S&P 500 over the period 1960-1996 and a standard deviation of realized dividend growth of 10%. The results are depicted in Figure 4.

7. To see this, note that

\[
E_t(x_s^\alpha) = E_t(\exp(\alpha \ln(x_s))) = \exp \left( \alpha E_t(\ln(x_s)) + \frac{1}{2} \alpha^2 \text{Var}_t(\ln(x_s)) \right).
\]

Now, using the fact that \( E_t(\ln(x_s)) = \ln(x_t) + \left( m_t - \frac{1}{2} \sigma^2 \right) (s - t) \) and \( \ln(E_t(x_s)) = \ln(x_t) + m_t (s - t) + \frac{1}{2} V_t (s - t)^2 \), we have \( E_t(\ln(x_s)) = \ln(E_t(x_s)) - \frac{1}{2} \left( \sigma^2 (s - t) + V_t (s - t)^2 \right) \). Hence, we can write

\[
E_t(x_s^\alpha) = (E_t(x_s))^\alpha \exp \left( \frac{1}{2} \alpha \text{Var}_t(\ln(x_s)) - \frac{1}{2} \alpha \text{Var}_t(\ln(x_s)) \right)
\]

\[
= (E_t(x_s))^\alpha \exp \left( -\frac{1}{2} \alpha (1 - \alpha) \text{Var}_t(\ln(x_s)) \right).
\]

Thus, the share price is given by the expression

\[
S_t = \frac{1}{e^{-\rho t} x_t^{\alpha}} E_t \left( \int_t^T e^{-\rho s} x_s^\alpha ds \right) = \frac{1}{e^{-\rho t} x_t^{\alpha}} \int_t^T e^{-\rho s} (E_t(x_s))^\alpha \exp \left( -\frac{1}{2} \alpha (1 - \alpha) \text{Var}_t(\ln(x_s)) \right) ds.
\]

When \( \alpha > 0 \), this expression will increase with the path of expected future dividends \( \{E_t(x_s)\} \) and fall with the path of future variance \( \{\text{Var}_t(\ln(x_s))\} \). When \( \alpha < 0 \), the opposite will occur.

8. Based on an analysis of the demand for risky assets, FRIEND and BLUME (1975) find that the average coefficient of relative risk aversion is probably well in excess of one and perhaps in excess of two. Using an analysis of deductibles in insurance contracts, DRÈZE (1981) finds even higher values.
As parameter uncertainty $V_t$ is increased, the price-earnings ratio rises. The effect is strongest for a long time horizon ($T = 25$, solid line) and decreases sharply as the time horizon is reduced ($T = 20$, dashed line; $T = 15$, dotted line). (Value of the parameters: $\alpha = -2$, $m_t = 0.06$, $\rho = 0.05$, $\sigma = 0.1$, $t = 0$, $T \in \{15, 20, 25\}$)

They show that especially for relatively long time horizons ($T = 25$), parameter uncertainty can have a sizable effect on the equilibrium price-earnings ratio. Moreover, the results demonstrate that, as the time horizon is reduced, the effect of a given level of parameter uncertainty on the P/E ratio declines as well. It is worth noting, however, that the results in Figure 4 underestimate the reduction in the effect of incomplete information on the equilibrium P/E ratio as time passes. The reason is that as time passes and more dividend information becomes available, the agent learns more about $\mu$ and his parameter uncertainty $V_t$ is reduced.

4.3 Parameter Uncertainty and Stock Price Dynamics

Besides leading to higher asset values, a high degree of parameter uncertainty $V_t$ has an influence on how stock prices react to changes in dividends. Applying Ito’s formula to (22), the dynamics of $S_t$ can be computed as

$$dS_t = \left( (1 - \alpha)m_t + \rho + \frac{1}{2} \alpha(1 - \alpha)\sigma^2 - \frac{1}{F(m_t, t)} + \alpha(1 - \alpha)V_t \frac{G(m_t, t)}{F(m_t, t)} \right) S_t dt$$
$$+ \sigma \left( 1 + \alpha \frac{V_t \frac{G(m_t, t)}{F(m_t, t)}}{\sigma^2 F(m_t, t)} \right) S_t d\tilde{B}_t$$
$$\equiv u_s S_t dt + \sigma_S S_t d\tilde{B}_t,$$

9. This effect is driven by the assumption that the true $\mu$ is constant through time. If one were to assume a time-varying $\mu$ instead, then $V_t$ would converge to a strictly positive limit and the effect of incomplete information on the equilibrium P/E ratio would no longer vanish as time passes.
where

\[ F(m_t, t) = \int_t^T \exp \left( \alpha \left( m_t - \frac{\rho}{\alpha} - \frac{1}{2} (1 - \alpha) \sigma^2 \right) (s - t) + \frac{\alpha^2}{2} V_t (s - t)^2 \right) ds > 0 \]  

(32)

and

\[ G(m_t, t) = \int_t^T (s - t) \exp \left( \alpha \left( m_t - \frac{\rho}{\alpha} - \frac{1}{2} (1 - \alpha) \sigma^2 \right) (s - t) + \frac{\alpha^2}{2} V_t (s - t)^2 \right) ds > 0. \]  

(33)

From equation (31), it is easy to see that whenever uncertainty about future growth in dividends exists \((V_t > 0)\), \(\sigma_S > \sigma\) if \(\alpha > 0\) and \(\sigma_S < \sigma\) if \(\alpha < 0\). This means that when the investor is less risk-averse than the log-utility investor, the price overreacts to changes in dividends. The opposite occurs when the investor is more risk-averse than the log-utility investor. In the special case of logarithmic utility \((\alpha = 0)\), the stock price changes in line with changes in the fundamental, the dividend process \(x_t\). Thus, in general, the degree of dividend growth uncertainty will not only influence the market value of capital assets. It will also have an effect on their reaction to dividend news.

Note that

\[ \frac{1}{\sqrt{V_t}} G(m_t, t) \]

follows itself a diffusion process. Thus, in this simple asset pricing model, asset price volatility under incomplete information is time-varying and stochastic whenever \(\alpha \neq 0\), i.e. whenever the representative agent doesn’t have logarithmic utility. Note that this effect arises in spite of the constant volatility \(\sigma\) of the fundamental, the dividend process \(x_t\). Under complete information, \(V_t = 0\) and price volatility is constant and equal to the underlying volatility of the fundamental, \(\sigma_S = \sigma\).

### 4.4 Hedging Demand and the Equilibrium Price of Estimation Risk

It is a well-known result of the theory of dynamic portfolio choice that investors will want to hedge against unfavorable random changes in their investment opportunity set. A question that naturally arises in the context of the model considered in this paper is whether such hedging demand is present and influences the equilibrium risk premium. From the analysis in Brennan (1998a) and Merton (1973), the investor’s optimal portfolio demand can be written as

\[ w = \frac{J_{WW}^{-1} \mu_S - r_t}{-J_{WW} W_t \sigma^2_S} + \frac{J_{Wm} V_t}{-J_{WW} W_t \sigma S}, \]

(35)
where \( J(W, m_t, t) \) denotes the investor’s indirect utility function, subscripts of \( J \) denote partial derivatives, \( \mu_S^C = (1 - \alpha)m_t + \rho + \frac{1}{2}\alpha(1 - \alpha)\sigma^2 + \alpha(1 - \alpha)V_t \frac{G(m_t, t)}{F(m_t, t)} \) is the cum-dividend expected return on the stock and \( r_t = \rho + (1 - \alpha)(m_t - \frac{1}{2}(2 - \alpha)\sigma^2) \) the risk-free interest rate.\(^{10}\) The first term in expression (35) is the investor’s tangency, the second component his hedging portfolio demand.

Under constant relative risk aversion, the value function \( J \) can be rewritten as the product of two functions, \( J(W, m_t, t) = \frac{W}{\alpha} I(m_t, t) \), and the optimal portfolio rule becomes\(^{11}\)

\[
w = \frac{1}{1 - \alpha} \mu_S^C - r_t + \frac{V_t}{(1 - \alpha)\sigma S} \frac{I_m}{I} . \tag{36}\]

Using the equilibrium values of \( \mu_S^C, \sigma_S \) and \( r_t \), the investor’s tangency portfolio demand can be rewritten as

\[
\frac{1}{1 - \alpha} \mu_S^C - r_t = \frac{1}{1 + \alpha} \frac{V_t}{\sigma^2} \frac{G(m_t, t)}{F(m_t, t)} . \tag{37}\]

Note that whenever the investor is less risk-averse than the log-utility investor \((\alpha > 0)\), the tangency portfolio weight is less than unity, implying that the investor’s hedging demand is positive. The reverse holds when the investor is more risk-averse than the log-utility investor \((\alpha < 0)\). The results in this paper are thus consistent with the analysis in Brennan (1998a), who demonstrates that investors that are less (more) risk-averse than the log-utility investor increase (decrease) their demand for the risky asset as the quality of their information about expected returns deteriorates.

It is worth noting that this hedging behavior influences the equilibrium normalized risk premium in the economy,

\[
\frac{\mu_S^C - r_t}{\sigma_S^2} = \frac{1 - \alpha}{1 + \alpha} \frac{V_t}{\sigma^2} \frac{G(m_t, t)}{F(m_t, t)} . \tag{38}\]

When the agent is less risk-averse than the log-utility investor \((\alpha > 0)\), the increase in his overall demand for the risky asset due to hedging leads to a decrease in the equilibrium risk premium below the value of \(1 - \alpha\) that would prevail under complete information. The reverse holds when the agent is more risk-averse than the log-utility investor \((\alpha < 0)\). Thus, the equilibrium price of dividend growth estimation risk is negative when \(\alpha > 0\) and positive when \(\alpha < 0\).

\(^{10}\) As shown in Chapter 10 of Duffie (1996), the equilibrium short rate \( r_t \) can be computed as

\[r_t = -\frac{\mu_x}{\sigma},\]

where \( \mu_x \) denotes the drift of the state-price deflator. Applying Ito’s formula to \( \pi_t = e^{-\mu t}x^\alpha t \) and using the dynamics of \( x_t \), the expression \( r_t = \rho + (1 - \alpha)(m_t - \frac{1}{2}(2 - \alpha)\sigma^2) \) follows.

\(^{11}\) See Brennan (1998a) and Merton (1973).
5. CONCLUSION

In the context of a representative-agent model, this paper analyzed the effect of dividend growth uncertainty or, in a somewhat broader interpretation, of macroeconomic uncertainty on stock prices. Our results illustrate that parameter uncertainty may have two important effects on stock prices. First, parameter uncertainty can lead to higher stock market valuations. This effect arises because, as uncertainty about the true expected growth in dividends in the economy increases, the expected value of future income streams rises. One could, of course, argue that this effect is driven by the parameterization of the model, in which uncertainty about growth rates (and not about levels) is assumed. This assumption, however, is not unreasonable. In practice, many if not most economists and analysts do not forecast levels, but growth rates. Moreover, many economic variables exhibit exponential growth. Assuming a geometric process for dividends is therefore justified.

Second, our analysis implies that parameter uncertainty influences the market price’s reaction to dividend news. The reason is that as new dividend information comes in, economic agents revise their estimates of future growth. The magnitude by which prices will under- or overreact to changes in dividends depends both on the degree of parameter uncertainty and on investors' risk aversion.

The analysis in this paper implies that current stock market valuations need not be the result of high expected future growth. Rather, they may arise because of uncertainty about future growth. Thus, one should be careful not to infer future economic growth based solely on current stock market valuations. This problem arises because it is not possible to determine both the expected growth in dividends and the degree of parameter uncertainty based on a single (stock) price. However, financial innovations may help address this issue. Consider, for example, a product such as the stripped dividend index proposed by Brennan (1998b). The introduction of this product would provide investors with a price for each year’s worth of dividends on the index instead of a single price summarizing all future dividends. This would have two main effects. First, it would allow investors’ diverse information about the future growth in dividends to be aggregated more efficiently and improve each individual investor’s quality of information, thus reducing parameter estimation risk at the economy-wide level and its effects on equilibrium prices and risk premia. Second, because expected dividend growth enters the valuation equation as a linear function of the time horizon while parameter uncertainty is inherently quadratic, one could in principle infer from the term structure of strip prices whether high valuations are caused by the level of expected dividend growth or by the uncertainty about it. Thus, financial innovations may not only improve the quality of information on the market. They may also help us resolve the present valuation puzzle.

12. I am grateful to the referee for pointing this out to me.
APPENDIX

Computing the Conditional Mean and Variance of $\ln(x_s)$

This appendix shows how to compute the conditional mean and variance of $\ln(x_s)$. Recall that $x$ has dynamics

$$dx_s = m_s x_s ds + \sigma x_s dB_s,$$  \hspace{1cm} (39)

and that

$$dm_s = \frac{V_s}{\sigma} dB_s.$$ \hspace{1cm} (40)

Applying Ito's formula yields

$$d\ln(x_s) = (m_s - \frac{1}{2} \sigma^2) ds + \sigma dB_s.$$ \hspace{1cm} (41)

Since $m$ is a martingale, using Fubini's theorem yields

$$E_t(\ln(x_s)) = \ln(x_t) + \int_t^s \left( m_u - \frac{1}{2} \sigma^2 \right) du = \ln(x_t) + \left( m_t - \frac{1}{2} \sigma^2 \right)(s - t),$$ \hspace{1cm} (42)

which is the expression given in the text. In order to compute the conditional variance of $\ln(x_s)$, let

$$Y_s = \begin{pmatrix} y_s \\ m_s \end{pmatrix} = \begin{pmatrix} \ln x_s + \frac{1}{2} \sigma^2(s - t) \\ m_s \end{pmatrix}.$$ \hspace{1cm} (43)

Then, by Ito's formula, we have$^{13}$

$$dY_s = \begin{pmatrix} dy_s \\ dm_s \end{pmatrix} = \begin{pmatrix} d\ln x_s + \frac{1}{2} \sigma^2 ds \\ dm_s \end{pmatrix} = \begin{pmatrix} m_s ds + \sigma dB_s \\ \frac{V_s}{\sigma} dB_s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} Y_s ds + \begin{pmatrix} \sigma \\ \frac{V_s}{\sigma} \end{pmatrix} dB_s.$$ \hspace{1cm} (44)

13. The change in the drift term is made for analytical convenience and has no effect on the conditional variance of $\ln(x_s)$. 
Let $Z(s)$ denote the conditional variance of $Y_s$, $Z(s) = \text{Var}_t(Y_s)$. Then, $Z(t) = 0$ and

$$
\begin{pmatrix}
    z_{11} & z_{12} \\
    z_{21} & z_{22}
\end{pmatrix} = \begin{pmatrix}
    0 & 1 \\
    0 & 0
\end{pmatrix} \begin{pmatrix}
    z_{11} & z_{12} \\
    z_{21} & z_{22}
\end{pmatrix} + \begin{pmatrix}
    z_{11} & z_{12} \\
    z_{21} & z_{22}
\end{pmatrix} \begin{pmatrix}
    0 & 0 \\
    1 & 0
\end{pmatrix} + \left( \frac{\sigma}{\sqrt{t}} \right) \left( \begin{pmatrix}
    \sigma \\
    \frac{\sqrt{t}}{\sigma}
\end{pmatrix}
\right) = \begin{pmatrix}
    2z_{21} + \sigma^2 & z_{22} + \sigma \\
    z_{22} + \sqrt{t}
\end{pmatrix}.
\tag{45}
$$

Hence, using the fact that $V_t = \frac{1}{\sqrt{t}}$, we have

$$
z_{22}(s) = \int_t^s \frac{V_u^2}{\sigma^2} \, du = \frac{1}{\sigma^2} \int_t^s \left( \frac{1}{\sqrt{t}} + \frac{1}{\sigma^2} u \right)^2 \, du = - \frac{1}{\sqrt{t}} \left[ \frac{1}{V_0 + \frac{1}{\sigma^2} u} \right]^s_t = \frac{1}{\sqrt{V_0 + \frac{1}{\sigma^2} t}} - \frac{1}{\sqrt{V_0 + \frac{1}{\sigma^2} s}}, \tag{46}
$$

and

$$
z_{12}(s) = z_{21}(s) = \int_t^s (z_{22}(u) + V_u) du = \int_t^s \left( \frac{1}{\sqrt{V_0 + \frac{1}{\sigma^2} u}} \right) du = \frac{s - t}{\sqrt{V_0 + \frac{1}{\sigma^2} t}}. \tag{47}
$$

and

$$
z_{11}(s) = \int_t^s (2z_{12}(u) + \sigma^2) du = \int_t^s \left( \frac{2u - t}{\sqrt{V_0 + \frac{1}{\sigma^2} t}} + \sigma^2 \right) du = \left( \frac{(u - t)^2}{\sqrt{V_0 + \frac{1}{\sigma^2} t}} + \sigma^2 \right) \bigg|_t^s = \frac{(s - t)^2}{\sqrt{V_0 + \frac{1}{\sigma^2} t}} + \sigma^2 \left( s - t \right). \tag{48}
$$

Thus, we have

$$
\text{Var}_t(m_s) = z_{22}(s) = V_t - V_s, \tag{49}
$$

$$
\text{Cov}_t(\ln(x_s), m_s) = z_{12}(s) = V_t(s - t), \tag{50}
$$

$$
\text{Var}_t(\ln(x_s)) = z_{11}(s) = \sigma^2(s - t) + V_t(s - t)^2, \tag{51}
$$

which is the result shown in the main body of the text.

14. See Bryson and Ho (1975), Section 11.4.
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SUMMARY

One of the striking phenomena in recent stock market history is the continuous rise in stock prices. Classical explanations for this phenomenon have argued that these apparently high valuations are either caused by measurement error in accounting data, or by high expected future growth in dividends. In this paper, it is shown that a high degree of uncertainty about expected growth in dividends can lead to an increase in stock prices. Moreover, dividend growth uncertainty can lead stock prices to over- or underreact to changes in dividends.

ZUSAMMENFASSUNG


RESUME

La hausse continuelle des prix des actions constitue un des phénomènes intrigants de l'histoire boursière récente. Ce phénomène a été expliqué soit par des erreurs de mesure dues aux règles comptables, soit par une croissance attendue élevée des dividendes. Cet article démontre qu'une incertitude élevée quant à la croissance future des dividendes peut conduire à une hausse des prix des actions. De surcroît, une telle incertitude peut conduire les prix à sur- ou sous-réagir à des changements de dividendes.